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MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Measurement of Magnetic Hysteresis

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Phil. Trans. R. Soc. Lond. A 1902 **198**, 33-104
doi: 10.1098/rsta.1902.0002

II. *The Measurement of Magnetic Hysteresis.*

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Communicated by Professor J. J. THOMSON, *F.R.S.*

Received May 2,—Read June 6, 1901.

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Introduction.

§ 1. Two ideal physical processes have been devised as the foundations of two methods of deducing mathematical expressions for the energy dissipated in magnetic material through magnetic hysteresis; these processes are due to Professor E. WARBURG and to the late Dr. J. HOPKINSON.

In WARBURG's theory* the specimen, in the form of a slender wire, is placed in a magnetic field due to a pair of permanent magnets so arranged as to produce magnetic force parallel to the length of the specimen. The mechanical work spent in moving these magnets through such a cycle of changes of position, that the iron is subjected to a cycle of magnetic changes, is clearly equal to the energy dissipated on account of magnetic hysteresis in the specimen. In terms of the magnetic quantities the energy dissipated per cub. centim. per cycle is $-\int I dH$ or $\int H dI$ ergs, where H is the magnetic force and I the intensity of magnetisation. Professor J. A. EWING† has applied the principle involved in WARBURG's theory to the design of a simple instrument by which the hysteresis of any specimen of sheet iron (for the range of induction $B = 4000$ to $B = -4000$ C.G.S. units approximately) is determined by comparison with two standard specimens supplied with the instrument, and previously tested for hysteresis by the ballistic method. The principle has also been employed by W. S. FRANKLIN,‡ by H. S. WEBB,§ and by G. L. W. GILL‖ to obtain absolute determinations of hysteresis.

The theory of the late Dr. JOHN HOPKINSON¶ proceeds in a different manner. The specimen now takes the form of a fine wire bent into a large circular ring, the ends of the wire being welded together; the length of the wire is l centim., and its cross-section A sq. centim. Let this ring be uniformly overwound with insulated wire at the rate of N turns per centim. so that the total number of turns is Nl, and let the wire be without resistance. Then if C be the current at any time, the magnetic force acting on the iron is $H = 4\pi NC$. If B be the magnetic induction in the iron, the number of linkages of lines of induction with the electric circuit at any instant is $AlNB$, and hence, when B changes, there is by FARADAY's law a voltage $AlN dB/dt$ between the ends of the coil. We have supposed here that the wire is closely wound upon the iron. The power spent in driving the current against this voltage is $AlNC dB/dt$ ergs per second.

Using the relation $H = 4\pi NC$, and noticing that Al is the volume (v) of the iron, the expression becomes $v/4\pi \cdot H dB/dt$.

* 'Wied. Ann.,' vol. 13 (1881), p. 141.

† 'Magnetic Induction in Iron and other Metals,' 3rd ed., revised, § 199.

‡ W. S. FRANKLIN, 'Physical Review,' vol. 2, p. 466.

§ H. S. WEBB, 'Physical Review,' vol. 8, p. 310.

‖ G. L. W. GILL, 'Science Abstracts,' vol. 1 (1898), p. 413.

¶ 'Phil. Trans.,' vol. 176 (1885), p. 466.

The work spent per unit volume during any finite change is thus

$$\frac{1}{4\pi}\int \mathrm{H}\frac{d\mathrm{B}}{dt}\,dt = \frac{1}{4\pi}\int \mathrm{H}d\mathrm{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

the expression found by HOPKINSON.

When the change is cyclic, so that B and H have the same values at the end as at the beginning of the cycle, we can throw the expression into a different form. For since $\mathrm{B} = \mathrm{H} + 4\pi\mathrm{I}$ we have $d\mathrm{B} = d\mathrm{H} + 4\pi d\mathrm{I}$. But when H goes (1) through the cycle $+\mathrm{H}_0, -\mathrm{H}_0, +\mathrm{H}_0$ or (2) goes from $+\mathrm{H}_0$ to $-\mathrm{H}_0$, then $\int \mathrm{H}d\mathrm{H} = 0$, and thus we recover WARBURG's expression $\int \mathrm{H}d\mathrm{I}$.

In the present paper we are only concerned with the work spent in causing a complete cycle of magnetic changes. We shall always use W to denote the energy dissipated by hysteresis per cub. centim. per cycle of magnetic changes, and we shall express W in ergs per cub. centim. per cycle. We have thus

$$\mathrm{W} = \frac{1}{4\pi}\int \mathrm{H}d\mathrm{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

To obtain the value of W by means of this expression, it has been usual to construct a cyclic B–H curve, best by the method described by EWING,* and to find its area. This process is easy enough, but since it involves the observations necessary to find at least ten or twelve points on the B—H curve, and the subsequent estimation of the area of the curve after it has been plotted, quite an hour is required for each determination of W.

§ 2. We have given the sketch contained in § 1 for the purpose of contrasting the physical ideas involved in the two mathematical methods by which the formulæ $\mathrm{W} = -\int \mathrm{I}d\mathrm{H}$ and $\mathrm{W} = 1/4\pi \,.\int \mathrm{H}d\mathrm{B}$ have been obtained, and of showing the manner in which the subject presented itself to one of us in 1895.

The remarks of § 1 refer only to *mathematical* processes and not to the experimental methods of studying the effects of hysteresis as exhibited in the relation of I or B to H. To the *experimental* knowledge of the subject the first contributions were made by the independent and nearly simultaneous papers of WARBURG and EWING. In addition to the theory noticed in § 1, WARBURG gave magnetometric observations of cyclic I—H curves, but his observations were few. EWING made a much more systematic attack on the subject, using the ballistic as well as the magnetometric method, and determining the values of $-\int \mathrm{I}d\mathrm{H}$ for a graded series of I—H curves for the same specimen of iron. An account of the subsequent development of the subject, in which EWING has had a great share, will be found in his book on 'Magnetic Induction in Iron and other Metals.' We owe much to Professor EWING, It was his hysteresis tester which formed the initial incentive to the research described

* EWING, 'Magnetic Induction in Iron and other Metals,' 3rd ed., revised, § 192.

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in the present paper, and, further, much of the general knowledge of magnetism needed in carrying out that research has been gained from his writings and from conversations with him.

§ 3. It occurred to one of us some years ago that just as EWING had, in effect, applied WARBURG's theory to produce a practical hysteresis tester, so it might be possible to apply HOPKINSON's theory to the design of a method which should give absolute determinations of the energy dissipated through hysteresis as quickly and as accurately as changes in magnetic induction are found by the aid of a ballistic galvanometer. It was evident that if this could be attained it would be possible to investigate the effects of various physical conditions—stress, temperature, the passage of an electric current, &c.—upon the hysteresis, with a comparatively very small expenditure of time. A preliminary account of the theory of the method was published in 1895,* and since then much time has been spent in working out some details which make the method practical; as only a few weeks in each year have been available for the work, progress has been slow.

In essentials the method is of course well known. In one of its forms it is in constant use among electrical engineers in testing by means of a watt-meter the energy dissipated in a transformer when its primary coil is traversed by an alternating current. In this case there is a *steady* deflexion of the watt-meter, and thus the watt-meter method is convenient in commercial work.

From the scientific standpoint, the watt-meter method has the disadvantage that it is not possible to find the limits between which the magnetic induction alternates without the use of revolving contact-makers, or oscillographs, or other appliances. The "effective" voltage is indeed easily measured, but unless the wave-form of the curve of voltage is known, the limits of the induction cannot be found.

With transformers of commercial dimensions the "effective" voltage is considerable, but when the iron is reduced to a single wire only 1 or 2 sq. millims. in section the "effective" voltage for 100 alternations per second does not exceed a few tenths of a volt, even when the secondary coil contains 1000 turns of wire. We know of no method by which so small an alternating voltage can be measured with any accuracy.

The idea which occurred to one of us in 1895 was to use a single reversal of the current to produce a "throw" of a ballistic electro-dynamometer instead of an alternating current to produce a steady deflexion of a watt-meter. The present paper contains an account of the development of this idea and of its applications to magnetic research.

So far as we know, the ballistic method of measuring hysteresis is novel. In the endeavour to make it a practical method, we have met with many difficulties, and the main part of the work has been devoted to overcoming those difficulties.

* G. F. C. SEARLE, "A Method of Measuring the Loss of Energy in Hysteresis," 'Proc. Camb. Phil. Soc.,' vol. 9, Part I., November 11, 1895.

Of the many advantages of the ballistic method, two may be mentioned. Thus the induction can be measured simultaneously with the hysteresis far more simply than when an alternating current is used. Further, the ballistic method enables measurements to be made so quickly as to render experiments easy which would otherwise be practically impossible on account of the very great time required for the numerous determinations of hysteresis necessary in investigations on the effects of stress or of temperature.

It will appear from the paper that several of the effects of physical changes upon *hysteresis* which we have studied presented themselves to us as we worked out the method. We found afterwards that some of these effects had already been discovered by EWING or others, or might have been deduced from their experiments. These cases we refer to in foot-notes. The effects of the various physical changes upon the *induction* have received so much attention from others that we have not thought it necessary to point out how much of that part of our work has consisted merely in going over old ground.

Approximate Theory of the Method.

§ 4. Let an iron ring of section A sq. centim. and mean circumference l centim. be wound with N turns of primary windings per centim., so that the total number of turns is Nl. The current C, which passes through the primary and magnetises the iron, producing the magnetic force $H = 4\pi NC$, passes also round the fixed coils of a sensitive electro-dynamometer. A secondary coil of n turns is wound over the iron, and is connected in series with the suspended coil of the dynamometer, and with an earth inductor, the total resistance of the secondary circuit being S. In finding the total induction through the secondary circuit, we must remember that the secondary will not generally be closely wound upon the iron. A certain number of lines of induction will in consequence pass through the secondary circuit due to the magnetic force H produced by the primary current. The total number of linkages of lines of induction with the secondary circuit is thus $AnB + MC$ where M is some constant.

Let the primary current C change gradually (1) from $+C_0$ to $-C_0$ and back again to $+C_0$, so as to complete a cycle, or (2) from $+C_0$ to $-C_0$, making a semi-cycle. During the change the voltage $AndB/dt + MdC/dt$ is set up in the secondary circuit. If the "time constant" of the secondary circuit be very small compared with the "time constant" of the primary circuit, the effect of the self-induction of the secondary circuit may be neglected, and the current in the secondary circuit may be taken to be

$$c = \frac{An}{S}\frac{dB}{dt} + \frac{M}{S}\frac{dC}{dt}.$$

Let the couple experienced by the suspended coil, when the currents in the fixed and suspended coils are C and c respectively, be $q\mathrm{C}c$ dyne-centims. Then since $\mathrm{H} = 4\pi\mathrm{NC}$, when the magnetic force due to the secondary current is negligible, we have for the couple at any instant

$$\text{Couple} = q\mathrm{C}c = q\frac{\mathrm{A}n}{4\pi\mathrm{NS}}\mathrm{H}\frac{d\mathrm{B}}{dt} + q\frac{\mathrm{M}}{\mathrm{S}}\mathrm{C}\frac{d\mathrm{C}}{dt}.$$

If the time of vibration of the moving coil be so great compared with the time occupied by the cycle or semi-cycle, that the cycle or semi-cycle is completed before the coil has sensibly moved from its equilibrium position, the angular momentum acquired by the coil is

$$\mathrm{K}\omega = \int q\mathrm{C}c\,dt = \frac{q\mathrm{A}n}{4\pi\mathrm{NS}}\int \mathrm{H}\frac{d\mathrm{B}}{dt}dt + \frac{q\mathrm{M}}{\mathrm{S}}\int \mathrm{C}\frac{d\mathrm{C}}{dt}dt \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where K is the moment of inertia of the coil, and ω is the angular velocity imparted to the coil by the electro-magnetic impulse. Now when C goes through either a cycle or a semi-cycle $\int \mathrm{C}d\mathrm{C}$ vanishes, and thus

$$\frac{1}{4\pi}\int \mathrm{H}d\mathrm{B} = \frac{\mathrm{NS}}{\mathrm{A}n}\int \mathrm{C}c\,dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Let θ be the greatest angular displacement or "throw" of the coil, and f the restoring couple exerted by the suspension per radian of displacement. Then by the principle of the conservation of energy, we may equate the initial kinetic energy to the potential energy at end of swing and thus obtain $\frac{1}{2}\mathrm{K}\omega^2 = \frac{1}{2}f\theta^2$,

or
$$\mathrm{K}^{\frac{1}{2}}\omega = f^{\frac{1}{2}}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

The three constants q, K, and f are eliminated, and the "constant" of the dynamometer is determined in the following manner. Let a constant current C′ flow in the primary circuit through the fixed coil, and while this current is passing, let the earth inductor be inverted so as to produce a known change, P, in the number of linkages of lines of induction with the secondary circuit. The time-integral of the current thereby produced is P/S, and hence the initial angular momentum of the suspended coil is

$$\mathrm{K}\omega' = q\mathrm{C}'\mathrm{P}/\mathrm{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

If ϕ be the throw produced, we have by (5) $\omega/\omega' = \theta/\phi$, and hence by (6) and (3),

$$\int \mathrm{C}c\,dt = \frac{\mathrm{C}'\mathrm{P}}{\mathrm{S}\phi}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Thus by (4)
$$\frac{1}{4\pi}\int \mathrm{H}d\mathrm{B} = \frac{\mathrm{NC}'\mathrm{P}}{\mathrm{A}n\phi}\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Since it is only the ratio of the angles θ and ϕ which appears in the formula for $1/4\pi \,.\, \int \mathrm{H}d\mathrm{B}$, we may take for the ratio θ/ϕ the ratio of the two "throws" of the spot of light along the scale, provided that the "throws" are not so large that $\tan 2\theta$ differs appreciably from 2θ.

The effect of damping has so far been neglected. If λ be the logarithmic decrement, we must write (5)

$$\mathrm{K}^{\frac{1}{2}}\omega = f^{\frac{1}{2}}\theta\,(1 + \tfrac{1}{2}\lambda),$$

where θ is now the *observed* "throw." The "throw" ϕ must be treated in the same way, and hence the ratio θ/ϕ is unaffected and the expression (8) remains true.

§ 5. If, instead of making the primary current go through a complete cycle, we make it go through a semi-cycle from $+ \mathrm{C}_0$ to $- \mathrm{C}_0$ or *vice versâ*, we can very conveniently combine the measurement of the hysteresis by the dynamometer with the measurement of the magnetic induction by a ballistic galvanometer. For if a ballistic galvanometer be connected in series with a second secondary coil wound upon the specimen, the throw of the galvanometer gives the value of $\mathrm{B}_1 - \mathrm{B}_2$, where B_1 and B_2 are the algebraical values of B corresponding to $+ \mathrm{C}_0$ and $- \mathrm{C}_0$. In the ideal case $\mathrm{B}_2 = - \mathrm{B}_1$, and then $\mathrm{B}_1 = \frac{1}{2}\,(\mathrm{B}_1 - \mathrm{B}_2)$.

Whether this ideal state obtain or not, we shall denote $\frac{1}{2}(\mathrm{B}_1 - \mathrm{B}_2)$ by B_0, calling it the mean maximum induction. When the specimen takes the form of a ring, the ballistic galvanometer only enables us to find B_0; it gives no information as to B_1 or B_2. This knowledge could, however, be obtained in the case of a long straight specimen by slipping the secondary coil off the specimen.

When, as in our case, there are two observers, the observations for W and B_0 can be made simultaneously.

§ 6. To take the specimen through a complete cycle, we must make the current go through the semi-cycle $+ \mathrm{C}_0$ to $- \mathrm{C}_0$ and then from $- \mathrm{C}_0$ to $+ \mathrm{C}_0$. If the iron has reached a cyclic state the throws of the galvanometer will be the same in magnitude, though opposite in direction for the two reversals of the current. But it will often happen that, on account of the previous treatment of the iron, B_1 differs considerably from $- \mathrm{B}_2$, and in this case the throws of the dynamometer differ in magnitude for the two reversals, though they are in the same direction.

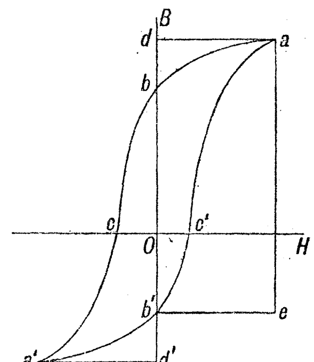

Fig. 1.

A little consideration will show us how to proceed in this case. Let $abca'b'c'a$ (fig. 1) be a B—H curve, and let ad, $a'd'$ be drawn parallel to OH. By reversal of the current, let the iron be caused to go through the changes represented by the curve $abca'$. From a to b H is positive and dB is negative, while from b to a' both H and dB are negative. Hence the value of $\int \mathrm{H}d\mathrm{B}$ for the semi-cycle $+ \mathrm{C}_0$ to $- \mathrm{C}_0$ is given by (area $bca'd'$) — (area abd). Similarly $\int \mathrm{H}d\mathrm{B}$ for the semi-cycle $-\mathrm{C}_0$ to $+\mathrm{C}_0$ is given by (area $b'c'ad$) $-$ (area $a'b'd'$). The value of $\int \mathrm{H}d\mathrm{B}$ for the whole cycle is thus the area of the B—H curve $abca'b'c'a$.

If θ_1 and θ_2 be the two throws of the dynamometer for the two semi-cycles, we have by (8)

$$\frac{1}{4\pi}\int_{\mathrm{B}_1}^{\mathrm{B}_2}\mathrm{H}d\mathrm{B} = \frac{\mathrm{NC'P}}{\mathrm{A}n\phi}\,\theta_1\,, \qquad \frac{1}{4\pi}\int_{\mathrm{B}_2}^{\mathrm{B}^1}\mathrm{H}d\mathrm{B} = \frac{\mathrm{NC'P}}{\mathrm{A}n\phi}\,\theta_2\,,$$

and hence, since W is the sum of the two quantities on the left sides of these equations,

$$\mathrm{W} = \frac{\mathrm{NC'P}}{\mathrm{A}n\phi}\,(\theta_1 + \theta_2)\,. \qquad \qquad (9).$$

Thus to find W we have simply to add together the two throws for the two semi-cycles, and then multiply the result by the factor $\mathrm{NC'P}/\mathrm{A}n\phi$.

The expression (9) is the fundamental formula of the method employed by us for the measurement of hysteresis. From it we see that all that is needed in addition to the electro-dynamometer for the measurement of W is an ampere-meter by which to determine C′, and an earth-inductor or some other means of producing a known change of induction (P) through the secondary circuit.

Complete Theory of the Method.

§ 7. In the elementary theory of § 4, the resistance, S, of the secondary circuit is supposed to be so large, and the induced current in consequence so feeble, that the magnetic force due to the induced current is negligible in comparison with that due to the primary current. It is thus necessary to proceed to a closer examination of the theory in order to find the correcting term which appears when S is only finite. In this examination we take account also of the energy dissipated by the eddy currents, which circulate in the specimen in consequence of the variations of the applied magnetic force. We use the notation already employed in the elementary theory.

§ 8. In the primary circuit let the magnetising coil and the dynamometer coil be placed next to each other in the circuit, and let E be the voltage at any instant between the ends of the conductor so formed. Now the induction through either circuit depends not only upon the magnetic induction, B, in the iron, but also upon the magnetic force in the space between the coils and the iron, as well as upon the magnetic force near the coils of the dynamometer. In each case the magnetic force depends only upon C and c, and not at all upon B, since the iron is formed into a ring. Thus if R be the resistance of the primary circuit between the two points between which the voltage is E, the equations for the primary and secondary currents may be written

$$\mathrm{E} = \mathrm{RC} + \frac{d}{dt}(\mathrm{N}l\mathrm{AB} + \mathrm{L'C} + \mathrm{M}c)\,. \qquad \qquad (10),$$

$$0 = \mathrm{S}c + \frac{d}{dt}(n\mathrm{AB} + \mathrm{MC} + \mathrm{L}c)\,. \qquad \qquad (11),$$

where B is the *average* value of the induction over the section of the specimen at any time. Here L and L′ are constant, while M varies slightly as the suspended coil turns round.

By the principle of the conservation of energy, the work done by the voltage E in any time is equal to the energy dissipated in the specimen, together with the heat produced in the resistances R and S, and the kinetic energy acquired by the moving coil of the dynamometer and the increase in the magnetic energy of the system.

Now let W_1, W_2 be the energy dissipated by hysteresis per cub. centim. for two semi-cycles, so that the loss per cycle is $W = W_1 + W_2$. Further, let X_1, X_2 be the space-averages* of the energy dissipated by eddy currents, so that the loss per cycle is $X = X_1 + X_2$. Then the total energy dissipated in the specimen in a semi-cycle is $Al(W_1 + X_1)$.

If ψ be the deflection of the suspended coil, then the couple tending to increase ψ is $CcdM/d\psi$ dyne-centim. The rate of working of this couple at any instant is thus $CcdM/d\psi . d\psi/dt$ or $CcdM/dt$, and hence the total work done is $\int CcdM/dt.dt$. This is therefore the kinetic energy acquired by the coil.

Then, if T, T′ be the magnetic energy at the beginning and end of a semi-cycle, we have

$$\int ECdt = (W_1 + X_1)\,Al + \int (RC^2 + Sc^2 + CcdM/dt)\,dt + T' - T.$$

But from (10)

$$\int ECdt = \int RC^2dt + \int C\frac{d}{dt}(NlAB + L'C + Mc)\,dt.$$

Comparing these expressions, we find, since $L'\int CdC/dt.dt = 0$ for a cycle or a semi-cycle,

$$(W_1 + X_1)\,Al = \int C\,(NlAdB/dt + Mdc/dt)\,dt - S\int c^2dt - T' + T.$$

The integrations are to be effected between the limits $t = 0$ and $t = \infty$, where $t = 0$ denotes any instant before the primary current begins to change, and $t = \infty$ denotes some instant towards the end of the change when the primary current has with sufficient accuracy reached its final value $\pm C_0$. It follows that $c = 0$ at both limits.

From (11) we find

$$-S\int c^2dt = \int c\frac{d}{dt}(nAB + MC)\,dt,$$

since $Lcdc/dt$ vanishes on integration. Adding this expression to the last one, we obtain

$$(W_1 + X_1)\,Al = \int \left[C\left(NlA\frac{dB}{dt} + M\frac{dc}{dt}\right) + c\frac{d}{dt}(nAB + MC)\right]dt - T' + T.$$

* We take the *space-average* because the rate of heat production is not uniform over the section; when the section is circular and dB/dt is very small the rate at any point is proportional to the square of the distance of the point from the centre of the section. (See Appendix I.)

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The terms in this formula which involve M are the complete differential of MCc, and thus vanish on integration, since $c = 0$ at both limits. We thus have

$$(W_1 + X_1)Al = \int CNlA \frac{dB}{dt} + \int c \frac{d}{dt}(nAB)\,dt - T' + T.$$

If we multiply (11) by NlC/n and integrate we obtain

$$\frac{NlS}{n}\int Cc\,dt = -\int C\left[NlA\frac{dB}{dt} + \frac{Nl}{n}\frac{d}{dt}(MC) + \frac{LNl}{n}\frac{dc}{dt}\right]dt.$$

Thus, by addition of the last two equations,

$$(W_1 + X_1)\,Al = -\frac{NlS}{n}\int Cc\,dt - \frac{Nl}{n}\int C\frac{d}{dt}(MC)\,dt + \int c\frac{d}{dt}(nAB)\,dt$$
$$- \frac{LNl}{n}\int C\frac{dc}{dt}\,dt - T' + T.$$

The first integral in this expression is proportional to the "throw" of the moving coil, for, by (7),

$$-\frac{NlS}{n}\int Cc\,dt = \frac{NlC'P}{n\phi}\,\theta_1,$$

if we measure θ_1 in the right direction.

As regards the second integral,

$$\int C\frac{d}{dt}(MC)\,dt = \Big[MC^2\Big]_0^\infty - \int MC\frac{dC}{dt}\,dt.$$

But since the time of vibration of the suspended coil is comparatively large, the change in C is practically complete before the coil has moved far from its equilibrium position. Hence M may be treated as constant for the whole range of integration, and thus, since $C^2 = C_0^{\,2}$ at both limits, the value of the integral is zero.

As regards the fourth integral,

$$-\int C\frac{dc}{dt}\,dt = -\Big[Cc\Big]_0^\infty + \int c\frac{dC}{dt}\,dt = \int c\frac{dC}{dt}\,dt,$$

since c vanishes at both limits.

Collecting these results, we have

$$(W_1 + X_1)\,Al = \frac{NlC'P}{n\phi}\,\theta_1 + \int c\frac{d}{dt}\left(nAB + \frac{LNl}{n}C\right)dt - T' + T.$$

The integral in this expression is the correction which makes its appearance when we take account of the finite conductivity of the secondary circuit. It will suffice to use in it the value of Sc which obtains when Ldc/dt is negligible in comparison with

$d(n\mathrm{AB} + \mathrm{MC})/dt$, viz., $\mathrm{S}c = -d(n\mathrm{AB}+\mathrm{MC})/dt$.* We may also treat M as constant. When this is done we obtain

$$(\mathrm{W}_1 + \mathrm{X}_1)\,\mathrm{A}l = \frac{\mathrm{N}l\mathrm{C'P}}{n\phi}\theta_1 - \frac{1}{\mathrm{S}}\int\left(n\mathrm{A}\frac{d\mathrm{B}}{dt} + \frac{\mathrm{LN}l}{n}\frac{d\mathrm{C}}{dt}\right)\left(n\mathrm{A}\frac{d\mathrm{B}}{dt} + \mathrm{M}\frac{d\mathrm{C}}{dt}\right)dt - \mathrm{T}' + \mathrm{T}.$$

If we add together the results for a pair of semi-cycles, the quantities T and T′ disappear when a cyclic state has been established, for then the magnetic energy is the same at the end as at the beginning of a cycle. We may then replace $\mathrm{W}_1 + \mathrm{X}_1$ by W + X and θ_1 by $\theta_1 + \theta_2$, but we must remember that the integral is to be taken completely round a cycle.

The expression can be put into a more convenient form it we notice that by the rules of approximation c is to be taken as zero in the terms under the integral sign. We thus replace $4\pi\mathrm{NC}$ by H, and then writing $4\pi\mathrm{N}d\mathrm{B}/d\mathrm{H}.d\mathrm{C}/dt$ for $d\mathrm{B}/dt$, we have, on division by al,

$$\mathrm{W} + \mathrm{X} = \frac{\mathrm{NC'P}}{\mathrm{A}n\phi}(\theta_1 + \theta_2) - \frac{\mathrm{N}}{\mathrm{SA}n}\int\left(\frac{4\pi n^2\mathrm{A}}{l}\frac{d\mathrm{B}}{d\mathrm{H}} + \mathrm{L}\right)\left(4\pi n\mathrm{AN}\frac{d\mathrm{B}}{d\mathrm{H}} + \mathrm{M}\right)\frac{d\mathrm{C}}{dt}d\mathrm{C}. \quad . \ (12),$$

the final formula.

Now, unless the specimen be so thick, and the variation of H so rapid, that during part of a complete cycle the *average* induction for the whole section increases while the applied magnetic force diminishes, $d\mathrm{B}/d\mathrm{H}$ is always positive. Hence, since $d\mathrm{C}/dt$ always has the same sign as $d\mathrm{C}$, the correcting integral is always positive. Thus the true value of W + X is less than that calculated from the throw of the dynamometer on the assumption that S is infinite.

§ 9. Before we can apply equation (12) in finding W, we must either know that X

* We can easily obtain with rough approximation the condition which must be satisfied in order that $\mathrm{L}dc/dt$ may be negligible in comparison with $d(n\mathrm{AB} + \mathrm{MC})/dt$. From (11) we have

$$\left(\mathrm{S} + \mathrm{L}\frac{d}{dt}\right)c = -\frac{d}{dt}(n\mathrm{AB} + \mathrm{MC}),$$

whence

$$-\mathrm{S}c = \mathrm{S}\left(\mathrm{S} + \mathrm{L}\frac{d}{dt}\right)^{-1}\frac{d}{dt}(n\mathrm{AB} + \mathrm{MC}) = \left(\frac{d}{dt} - \frac{\mathrm{L}}{\mathrm{S}}\frac{d^2}{dt^2}\ldots\right)(n\mathrm{AB} + \mathrm{MC}).$$

Thus the approximation $-\mathrm{S}c = d(n\mathrm{AB} + \mathrm{MC})/dt$ is valid if $\mathrm{L/S}\,.\,d^2(n\mathrm{AB} + \mathrm{MC})/dt^2$ is small in comparison with $d(n\mathrm{AB} + \mathrm{MC})/dt$. A sufficient idea of the magnitudes involved is obtained by treating B as proportional to C. In this case $\mathrm{L/S}\,.\,d^2\mathrm{C}/dt^2$ must be small in comparison with $d\mathrm{C}/dt$. But, if E denote the voltage of the battery, the characteristic of C is

$$\mathrm{K}d\mathrm{C}/dt + \mathrm{RC} = \mathrm{E},$$

where R is now the resistance of the whole of the primary circuit, and K depends mainly upon the choking coil (§ 33) in the circuit. Hence, treating K as constant,

$$\mathrm{K}d^2\mathrm{C}/dt^2 + \mathrm{R}d\mathrm{C}/dt = 0.$$

Thus the condition for the validity of the approximation can be expressed in simple form by saying that L/S must be small in comparison with K/R.

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is negligible or else be able to determine it. In most of our experiments the specimens have been fine wires, and X has been insignificant in comparison with W, but with rings of solid metal, such as those used by Mr. R. L. WILLS,* with a sectional area of over 1 sq. centim., X becomes of real importance. Now from the behaviour of the eddy currents in a rod of circular section, we may assert that when the "time constant" (inductance/resistance) of the primary circuit is large compared with $\pi\mu a^2/\sigma$, where a is the radius of the largest circle inscribable in the section, and μ and σ are the permeability and specific resistance of the material, then the eddy current at any point may be calculated on the assumption that the magnetic induction has at any instant the same value at all points of the section (Appendix I.). In this case the eddy current at any given point is proportional to $d\text{B}/dt$, and hence, as the method of "Dimensions" shows, the *space-average* of the rate at which heat is generated per unit volume may be written

$$d\text{X}/dt = \text{QA}\,(d\text{B}/dt)^2/\sigma = 16\pi^2\text{N}^2\text{AQ}/\sigma\,.\,(d\text{B}/d\text{H})^2\,.\,(d\text{C}/dt)^2 \quad . \quad . \quad . \quad (13),$$

where Q is a constant depending upon the *form* of the section. On reference to the meaning of X it will be seen that the total rate of heat production by eddy currents per unit length of the specimen is $\text{QA}^2(d\text{B}/dt)^2/\sigma$. For a circular section $\text{Q} = 1/8\pi = \cdot 03979$, and for a square section $\text{Q} = \cdot 03512$. (Appendix I.)

We can now write (12) as follows :—

$$\begin{aligned} \text{W} &= \frac{\text{NC}'\text{P}}{\text{A}n\phi}(\theta_1 + \theta_2) \\ &\quad - \int\left[\frac{16\pi^2\text{N}^2\text{QA}}{\sigma}\left(\frac{d\text{B}}{d\text{H}}\right)^2 + \frac{\text{N}}{\text{SA}n}\left(\frac{4\pi n^2\text{A}}{l}\frac{d\text{B}}{d\text{H}} + \text{L}\right)\left(4\pi n\text{AN}\,\frac{d\text{B}}{d\text{H}} + \text{M}\right)\right]\frac{d\text{C}}{dt}\,d\text{C} \\ &= \text{U} - \text{X} - \text{Y} \quad (\S 13) \quad . \quad . \quad . \quad . \quad . \quad . \quad (14). \end{aligned}$$

If the specimen be built up of p similar wires so that the total cross-section is A, then X is $1/p$ times the value for a single wire of section A. The p wires must be insulated from each other so that there are no eddy currents from wire to wire.

The quantity $d\text{B}/d\text{H}$, which occurs in the correcting integral, is for a given specimen a nearly definite (double valued) function of H and therefore of C, for given limits $\pm\,\text{C}_0$, provided that the "time constant" of the primary circuit is large compared

* Mr. R. L. WILLS, of St. John's College, 1851 Exhibition Scholar, began in 1900, at the Cavendish Laboratory, a series of experiments on the effect of temperature upon the energy dissipated through hysteresis in iron and alloys of iron, in continuation of his work on the "Effects of Temperature on the Magnetic Properties of Iron and Alloys of Iron" ('Phil. Mag.,' July, 1900). At the suggestion of Professor J. J. THOMSON he employed the method described in the present paper, while we gladly furnished him with some of the apparatus employed by us in our own researches. It was plain that in the specimens used by Mr. WILLS the energy dissipated by eddy currents was comparatively much greater than in our own specimens, and so might be far from negligible. We were thus led to extend the theory so as to take account of the eddy current loss, and to devise a method of determining that loss.

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with $\pi\mu a^2/\sigma$. (Appendix I.) Quite apart from the effects of eddy currents, dB/dH appears to depend slightly upon the rate of variation of H, but so slightly that we may for our present purpose consider dB/dH as having two definite values for any given value of C when C varies between given limits.

Hence for a given specimen, the correction to be subtracted from the value of W calculated from the throw of the dynamometer by the elementary theory will be increased p-fold if, for every value of C which occurs in the cycle, dC/dt be increased p-fold. The part of the correction involving the resistance, S, of the secondary circuit, is inversely proportional to S. It also depends upon L and M and diminishes with those quantities.

When the secondary coil contributes practically the whole of L and M we have approximately

$$L = 4\pi n^2 (G - A)/l, \qquad M = 4\pi N n (G - A) \quad . \quad . \quad . \quad . \quad . \quad (15),$$

where G is the mean area of one turn of the secondary coil.

In this case, unless G is many times greater than A, L and M are negligible in comparison with the quantities added to them in (14), since for iron dB/dH is generally very large.

Referring to the definition of L and M in § 8, we see that as far as these quantities are concerned the correction is made as small as possible by winding the secondary windings as closely as possible upon the iron, and by making the self-induction of the rest of the secondary circuit as small as possible. To secure the latter point the earth inductor employed to standardise the dynamometer should be removed from the secondary circuit, and a non-inductive coil of equal resistance inserted in its place when the dynamometer readings for W are taken.

§ 10. For different specimens of iron the correction will depend upon dB/dH. Indeed since L and M can be made comparatively small, the correction depends mainly upon the *square* of dB/dH.

It is, of course, impossible to assign any definite value to dB/dH for any particular specimen, for the ratio varies very greatly for different parts of the B—H curve. All we can say is that if for any part of the B—H curve dB/dH has a very high value, and if for the corresponding value of C the rate dC/dt is not very small, then this portion of the B—H curve may make a considerable contribution to the correcting integral in (14).

Now the maximum value of dB/dH varies greatly in different kinds of iron, being small in steel and very large in good soft iron. According to EWING's experiments in the case of considerable magnetic forces, the maximum value of dB/dH for glass-hard steel is about 300, while for soft-iron wire, suitable for the manufacture of transformers, its value rises to 13,000. The maximum value of $(d\text{B}/d\text{H})^2$ for the iron is accordingly about 2000 times its value for the steel. It is thus evident that though the method may give fairly accurate results for steel without any special precautions to secure that dC/dt should never be large, it may give quite inaccurate results for soft iron unless the proper precautions are taken. A considerable portion of the time spent upon this research was occupied in investigating the precautions necessary to secure accuracy.

§ 11. If we consider the effect of changing the area of section of the iron we shall see that the part of the correction due to the conductivity of the secondary circuit increases as the section diminishes. For neglecting L and M, (14) becomes

$$W = \frac{NC'P}{An\phi}(\theta_1 + \theta_2) - 16\pi^2 N^2 A \left(\frac{Q}{\sigma} + \frac{n^2}{Sl}\right) \int \left(\frac{dB}{dH}\right)^2 \frac{dC}{dt} dC. \quad . \quad . \quad . \quad (16).$$

Now for a given kind of iron, if we wish to obtain the same throw of the dynamometer, when we halve A we must double n. Thus we must regard An as determined by the sensitiveness of the dynamometer. Hence the part of the correction which depends upon S is proportional to n, and therefore inversely proportional to A. This explains the difficulty we met with in the earlier stages of the work. We were able to get accurate results for specimens built up of fine wires, with a total section of 1 sq. centim., though we failed to do so for specimens with a section of 1 or 2 sq. millims.

The part of the correcting term in (16) depending upon S is inversely proportional to l. The circumference of the ring should therefore be large. When, as in most of our experiments, a straight wire of finite length is used, the secondary coil should be wound upon a long bobbin and not be heaped up at one part of the wire.

§ 12. *Effect due to Air alone.*—If there be no iron in the system $B = H$ everywhere, and $W = 0$; if, further, there be no metal, *e.g.*, brass, inside either the primary or the secondary coil $X = 0$ also. In this case (14) reduces to

$$\frac{C'P}{\phi}(\theta_1 + \theta_2) = \frac{1}{S}\left(\frac{4\pi n^2 A}{l} + L\right)(4\pi nAN + M)\int \frac{dC}{dt} dC \quad . \quad . \quad . \quad . \quad (17).$$

Since dt is necessarily positive, dC/dt has the same sign as dC, and hence the integral cannot vanish. There will, therefore, always be a small throw of the dynamometer even when there is no metal in the coils. But since dB/dH is always positive and is generally large, it is easily seen that unless L and M (§ 8) are very large in comparison with the quantities to which they are added in (17), the throw, when the iron is out, is small compared with the part of the throw which is due to the correcting integral in (14), when the iron is in. This deduction we verified by experiment, for in many trials we never found any case in which we could detect any throw when there was no metal in the magnetising coil.

Determination of the Correction for the Finite Conductivity of the Secondary Circuit, and of the Energy dissipated by Eddy Currents.

§ 13. It is convenient to have a single symbol for the quantity $NC'P(\theta_1 + \theta_2)/An\phi$, and we shall use U for this purpose. Thus U is the value of the hysteresis loss which is calculated from the throw of the dynamometer coil simply, without any regard to the corrections.

If, further, we denote by Y the correction due to the finite conductivity of the secondary circuit, we can write (14) in the form

$$W = U - (X + Y) = U - Z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

where $Z = X + Y.$

Comparing (18) with (14), we see that we can put

$$X = \frac{1}{\sigma}\int x \frac{dC}{dt}\, dC, \qquad Y = \frac{1}{S}\int y \frac{dC}{dt}\, dC \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

where x and y depend only upon C, and not upon dC/dt.

§ 14. The form of Y at once suggests the way to find Y. For if U, Y correspond to S, and U′, Y′ to S′, then since X and W are unaffected by the change in S, $U - U' = Y - Y'$. But $SY = S'Y'$, and hence

$$Y = (U - U')\, S'/(S' - S) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Thus, by observing the values of U found in two experiments with two values for S, the values of Y and Y′ can be determined.

In this determination it is not necessary to find the throw for the earth inductor for each value of S. If there were no damping we should have $\phi' = S\phi/S'$, so that ϕ' could be calculated if ϕ were known for the one resistance S. But the logarithmic decrement depends upon S, being of the form $\lambda = u + v/S$, where v is proportional to the current C′, and hence each throw must be corrected for damping before it is used in the calculations. The easiest way of making the correction is to add to each throw a quarter of the difference between it and the next elongation on the same side when the dynamometer coil is allowed to continue in vibration. Since the throws are of very different magnitudes, it is necessary to correct each one for the difference between $\frac{1}{2}\tan 2\theta$ and θ, if accuracy be desired.

§ 15. The correction X is of great importance when the section of the specimen is of considerable area. In this case the secondary coil will generally be wound so closely upon the iron that L and M will be negligible in comparison with the quantities to which they are added in (14). Under these conditions the value of X/Y can be calculated, for now by (16) $X/Y = QSl/n^2\sigma$, and hence, when $X + Y$ is known, X can be calculated; for iron, $\sigma = 10^{-5}$ ohm per centimetre cube approximately. It is not practicable to calculate X from the value of Y found by varying S, for an examination of the numerical magnitudes shows that X/Y is now a large numeric—in Mr. WILLS's research something like 1000—while experiment shows that Y is so small as to elude observation.

To determine $X + Y$ or Z, when the section of the specimen is considerable, we must vary dC/dt in a known manner. This could be done in the following way:—

Let the poles of a battery of voltage E be joined to the points F, G (fig. 2), and let the resistance between F and G through the battery be T. Let F and G be also connected through the primary coil, the fixed coils of the dynamometer, and a

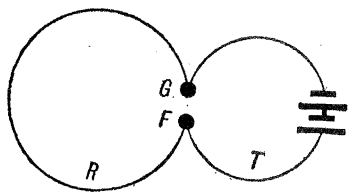

Fig. 2.

choking coil, the resistance of this part being R. Let the steady current, when F and G are not in contact, be C_0. Now let F and G be put into contact. The current in the primary coil then has the characteristic

$$K\, dC/dt + RC = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

where K depends upon C, since it involves the value of dB/dH for the core of the choking coil (§ 33).

After the current has sunk from C_0 to some very small value, let F and G be separated again, but before the separation let E be reversed. The characteristic is now

$$K\, dC/dt + (R + T)\, C = - E = - (R + T)\, C_0 \quad . \quad . \quad . \quad . \quad (22).$$

The current now gradually attains the value $-C_0$. The whole process is practically complete in a fraction of the time of vibration of the dynamometer coil.

The key described in § 32 carries out this process exactly, provided that the resistances D are made zero. We find that the time for which no E.M.F. acts on the circuit through the primary coil is about $\frac{1}{4}$ second, while a rough calculation shows that with our apparatus the current sinks very nearly to zero in that time.

Suppose now that E, R and T are all increased p-fold. Then C_0 remains unchanged, and for any given value of C, during the whole time of variation, dC/dt is also increased p-fold. Thus, if the ratio R/T be kept constant as well as the maximum current C_0, dC/dt for any value of C is proportional to E. Hence

$$Z = X + Y = zE \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23),$$

where z is a quantity independent of E. A method of finding Z similar to that employed for Y is now available. For if U, Z correspond to E, and U′, Z′ to E′, then $U' - U = Z' - Z$, since W remains unaltered, *unless the hysteresis depends upon the speed at which the magnetic changes occur.* But $Z/E = Z'/E'$, and hence

$$Z = (U' - U)\, E/(E' - E) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

Thus, by observing the values of U found in two experiments with two values for E, the values of Z and Z′ can be determined.

Practical examples of these two methods are given in §§ 36, 41, 42.

Application of the Method to Rods of Finite Length.

§ 16. In many experiments it is convenient, and in most of our experiments it was necessary, that the specimen should be a straight rod of finite length instead of a ring. We must therefore consider what modification of the theory is necessary in order to make it fit this case.

The magnetising solenoid will not be infinite in length, but the correction due to the finite length is very small when the diameter is small compared with the length. For if the mean radius of the windings is r, and the whole length of the solenoid is $2l$, then at a point within the solenoid whose distances from the central plane and from the axis are x and y, the components of the magnetic force are

$$H_x = 4\pi NC \frac{l}{\sqrt{l^2 + r^2}} \left\{1 - \tfrac{3}{2} \frac{r^2 (x^2 - \frac{1}{2}y^2)}{(l^2 + r^2)^2} - \ldots\right\} \quad \ldots \quad (25),$$

$$H_y = 4\pi NC \frac{l}{\sqrt{l^2 + r^2}} \left\{\tfrac{3}{2} \frac{r^2 xy}{(r^2 + l^2)^2} + \ldots\right\} \quad \ldots \quad (26).$$

The magnetic force is thus very nearly constant in the central parts of the solenoid, and it is practically sufficient to substitute for N in the expression $4\pi NC$ the quantity $N' = Nl(l^2 + r^2)^{-\frac{1}{2}}$.

In our experiments we had $l = 24$, $r = 2$ approximately, and thus the magnetic force at any point within the central 20 centims. of the solenoid did not differ from that at the centre by more than $\frac{1}{5}$ per cent.

§ 17. When the specimen is a finite straight rod instead of a ring, "poles" are developed upon it, and these give rise to a demagnetising force, h, thus causing the magnetic force at the centre of the rod to differ from that calculated from the currents in the primary and secondary circuits. As we are now dealing with a correction it will suffice to find the effect of the demagnetising force on the assumption that the elementary theory is applicable so that the magnetic force due to the secondary current is negligible. We further suppose that the secondary coil, which is placed round the centre of the rod, is short compared with both solenoid and rod; we can then treat the demagnetising force, h, as well as the magnetic force, $4\pi N'C$, due to the solenoid, as constant within the secondary coil. The rod may be either longer or shorter than the solenoid.

Let A be the area of the section of the iron and G the mean area of one turn of the secondary coil. We only restrict A to be constant in the neighbourhood of the secondary coil; the section may, if convenient, increase or decrease considerably at a distance from the secondary coil. If H be the magnetic force at the centre of the specimen, we have

$$H = 4\pi N'C - h \quad \ldots \quad (27).$$

The secondary current is given by

$$\mathrm{S}c = \mathrm{A}n\, d\mathrm{B}/dt + (\mathrm{G} - \mathrm{A})\, n d\mathrm{H}/dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

Substituting for C from (27) we find

$$4\pi\mathrm{N}'\mathrm{S} \int \mathrm{C}c dt = \mathrm{A}n \int (\mathrm{H} + h)\, d\mathrm{B} + (\mathrm{G} - \mathrm{A})\, n \int (\mathrm{H} + h)\, d\mathrm{H}.$$

Remembering that $\int \mathrm{H}d\mathrm{H} = 0$ for a cycle or a semi-cycle, we have, on using (7),

$$\frac{\mathrm{C}'\mathrm{P}}{\phi}\,\theta = \mathrm{S}\int \mathrm{C}c dt = \frac{\mathrm{A}n}{4\pi\mathrm{N}'}\left\{\int \mathrm{H}d\mathrm{B} + \int h d\mathrm{B} + \frac{\mathrm{G} - \mathrm{A}}{\mathrm{A}}\int h d\mathrm{H}\right\} \quad . \quad . \quad . \quad (29).$$

Hence the throw of the dynamometer is larger than that corresponding to $\int \mathrm{H}d\mathrm{B}$, and thus if we calculate W from the throw of the dynamometer we must subtract a correction depending upon the integrals $\int h d\mathrm{B}$ and $\int h d\mathrm{H}$. The correction can be found only when h is known both as a function of B and also as a function of H.

§ 18. Lord RAYLEIGH* was the first to show how to correct an I—H curve for the demagnetising force. He supposes that the specimen is an ellipsoid, and that the applied magnetic force is constant throughout its volume; the magnetic quantities H, B, I, and h are, in this case, constant throughout the specimen, and thus we can write $h = p\mathrm{I}$, where p is a constant factor (usually denoted by N). This relation between H and I allows the I—H curve for an infinitely long ellipsoid to be deduced from the curve for a short ellipsoid by "shearing" through a distance everywhere proportional to I.

Some investigators, supposing that the demagnetising force, h, at the centre of a long cylinder is the same as that for the ellipsoid inscribable in the cylinder, have applied Lord RAYLEIGH'S construction to the case of long cylinders. Others, such as Dr. H. DU BOIS,† though avoiding this error, have assumed that h can be expressed in the form $h = p\mathrm{I}$, where p is a function of the ratio of the length to the diameter, but is independent of I, the intensity of magnetisation at the centre of the cylinder. But it is easy to see that, quite apart from the influence of hysteresis, p cannot be constant, since the permeability of iron is not independent of the magnetic force. For, on account of the demagnetising action of the ends of the rod, the magnetic force near the ends differs from that near the centre of the rod, and thus the rod has, in effect, different values of μ in different parts. If μ were constant for each part, p would still be constant, but in the actual case, when the applied magnetic force is

* "On the Energy of Magnetised Iron," 'Phil. Mag.,' 1886, vol. 22, p. 175, or 'Scientific Papers,' vol. 2, art. 139.

† 'The Magnetic Circuit in Theory and Practice,' p. 41. Dr. DU BOIS, however, describes (p. 123) the experiments of LEHMANN upon the magnetisation of a toroid with a radial slit, and points out that LEHMANN'S results show that the demagnetising factor increases, but only gradually, as I increases up to about half its maximum value; beyond this point the increase is more rapid. The experiments were not arranged so as to show the effects of hysteresis.

changed, the value of μ for each part of the rod changes, the change being greater for some parts of the rod than for others. The magnetism which appears at any part of the rod is thus not proportional to the value of I at the centre of the rod, and accordingly p is not constant. The curves obtained by Mr. C. G. LAMB* show to how great an extent the distribution of magnetism depends upon the applied magnetic force.

But, further, if the rod be put through cycles of magnetic changes, as the ends of the rod are approached the range of the magnetic force changes, on account of the demagnetising action of the ends. Hence, since the behaviour of iron in regard to hysteresis depends upon the range of the magnetic force, the demagnetising force, h, at the centre does not depend simply upon the intensity of magnetisation, I, at the centre of the rod, but depends also upon the manner in which that value of I has been reached. Thus h will exhibit hysteresis with respect to I and hence also to B and H, each quantity referring to the centre of the rod.

§ 19. We give in Appendix II. a simple experimental method of determining h both as a function of B and as a function of $4\pi N'C$, the magnetic force due to the solenoid. Since $\int h dh = 0$ we may replace H by $4\pi N'C$ in the second correcting integral in (29). When h has been found in terms of B and H by this method, the values of $\int h d\text{B}$ and $\int h d\text{H}$ can be found by measurement of the areas of the h—B and h—H curves. Now in (29) the integral $\int h d\text{H}$ has the factor (G—A)/A, and hence disappears when G = A, *i.e.*, when the secondary coil is wound infinitely closely upon the iron. In most cases it will be convenient that the secondary coil should be wound upon a tube large enough to pass easily over the rod, and G will then be considerably larger than A, though G/A need not exceed 10 or 20 except for very thin rods. Now for a given step dB the corresponding step dH is always by comparison small, since $d\text{H}/d\text{B}$ is very small—perhaps 1/10000—in the steep parts of the cyclic B—H curve, and never rises above 1/100 unless the iron is well "saturated." Hence the term $(\text{G—A})/\text{A} \,.\int h d\text{H}$ will generally be negligible in comparison with $\int h d\text{B}$.

For a bundle of ten iron wires 47 centims. long and ·0412 sq. centim. in total area of section, placed in a solenoid 47 centims. long, and furnished with a secondary for which G = ·785 sq. centim, so that (G—A)/A = 18, we found (§ 78)

$$\int \text{H}d\text{B} = 89200\,, \quad \int h d\text{B} = 949\,, \quad \frac{\text{G} - \text{A}}{\text{A}}\int h d\text{H} = 1{\cdot}57 \times 18 = 28{\cdot}2\,.$$

In this case the second integral introduces a correction of about 1 per cent., and the third one is negligible. The limits of B were $\pm$ 9450, and those of H $\pm$ 10·65. The value of h at these limits was $\pm$ ·106.

* "On the Distribution of Magnetic Induction in a Long Iron Bar," 'Proc. Physical Society,' vol. 16, p. 509, or 'Phil. Mag.,' September, 1899.

For a single wire 47 centims. long and ·00412 sq. centim. in section, placed in a solenoid 60 centims. long, we had (G—A)/A = 37, and found (§ 79),

$$\int \mathrm{H}d\mathrm{B} = 102100\,, \quad \int hd\mathrm{B} = 72{\cdot}3\,, \quad \frac{\mathrm{G}-\mathrm{A}}{\mathrm{A}}\int hd\mathrm{H} = {\cdot}118 \times 37 = 4{\cdot}36\,.$$

The limits of B were ± 9850, of H ± 10·67, and of h ± ·0084.

§ 20. These corrections are also applicable to the area of a cyclic B—H curve drawn by the aid of a ballistic galvanometer. With respect to the maximum induction B_0, when the current C_0 is reversed, the throw of the galvanometer measures $AB_0 + (G-A)\,H_0$, where it will be sufficient to take H_0 as $4\pi N'C_0$, since $(G-A)\,H_0$ is small compared with AB_0. But in finding the maximum magnetic force H_0, the correction h_0 may not be negligible, and must be subtracted from the quantity $4\pi N'C_0$. The value of h_0 corresponding to H_0 is easily found by the method described in Appendix II.

Description of the Apparatus.

§ 21. We now pass on to describe the instruments employed in the experiments. It will perhaps conduce to clearness if we leave till last the description of the special appliances which were designed in order to secure accuracy.

The Ampere-Meter.

§ 22. In all except the preliminary part of the work a Weston direct reading ampere-meter was used for the measurement of the primary current. The instrument only read up to 1·5 amperes, but when the largest number of windings on the solenoid (§ 24) was employed this current gave a magnetic force of about 108 C.G.S. units. The instrument reads in one direction only, and thus it is necessary to join one of its terminals directly to one pole of the battery, so that the current always flows through it in the same direction.

The Earth Inductor.

§ 23. The earth inductor, which served throughout our experiments to produce a standard change of induction, was a simple coil of 100 turns, having a mean radius of about 18·2 centims.; its resistance is 1·345 ohms. Using a Clark cell as a standard of E.M.F., the change in the number of linkages of lines of induction with the circuit when the coil is turned over through 180° about a horizontal axis was found to be $8{\cdot}72 \times 10^4$ C.G.S. units.

We prepared a resistance coil of copper wire with the same resistance as the earth inductor. We could thus substitute this coil for the earth inductor when we desired, for the reason given in § 9, to get rid of the self-induction of the earth inductor and at the same time to keep the resistance of the circuit unchanged. Since both coils were of copper, their resistances were equal when their temperatures were equal.

The Magnetising Solenoid.

§ 24. In our experiments the specimens of iron have taken the form of straight wires, and in consequence the magnetising coil has been a straight solenoid. We used a straight wire instead of a ring in order to be able to apply tension or torsion to the specimen. The solenoid is formed of several independent layers of wire wound upon an ebonite tube 47 centims. in length. It is essential that the tube should be of non-conducting material, for otherwise the currents induced in it would cause the magnetic force to differ considerably from that calculated from the formula $H = 4\pi NC$; this effect would cause a considerable error in the measurement of the hysteresis. By combining the independent layers in different ways we could vary the magnetic force due to unit C.G.S. current by steps of about 50 from 50·55 to 718·6 C.G.S. units. The magnetic forces due to unit current in each of the four coils are as follows :—

AA, 50·55, BB, 50·55, XY, 212·2, MN, 405·1.

The resistances of the coils are ·54, ·54, 2·00, and 4·85 ohms respectively.

The Secondary Coils.

§ 25. The secondary coils were formed of fine insulated copper wire wound on narrow tubes of ebonite or glass, through which the specimen passed. Insulating material was used for the bobbins to avoid the induction of currents in them. In our earliest experiments it was found that metal tubes caused very large errors by the action of the currents induced in them. Several coils were used with the dynamometer, the number of windings varying from 300 to 1285; the coil of 1285 turns had a bobbin 11·5 centims. long, thus conforming to the recommendation of § 11. The mean area of this coil was ·785 sq. centim.

The Ballistic Galvanometer.

§ 26. The ballistic galvanometer was made by one of us, with some assistance from Mr. W. G. PYE. The magnet system consists of two vertical magnetised wires, each about 11·3 centims. long and 1 millim. in diameter. These are fixed parallel to each other at a distance of 1 centim., the north pole of one magnet being opposite the south pole of the other. When parallelism is secured, the magnetic system is necessarily astatic, however much the magnets may vary in strength. The system is suspended from a torsion head by a single phosphor-bronze wire $\frac{1}{1000}$ inch in diameter and 12 centims. in length. The torsion of this bronze wire supplies the restoring couple. There are four coils, each containing about 250 turns of No. 18 B.W.G. copper wire, the total resistance of the coils in series being 2·14 ohms. The coils are arranged so that the upper end of the magnet system is at the centre of the upper pair of coils and the lower end is at the centre of the lower pair of coils. The motion of the magnets is observed by the aid of a lamp and scale.

The galvanometer is very sensitive, and thus measurements of B for quite thin iron wires can be made with a comparatively small number of turns of secondary winding; this is often a point of some convenience.

We have found the instrument very efficient. The time of vibration, 12 seconds, is long enough to enable the ballistic throws to be read with ease. Since the needle is practically astatic, the zero depends only on the action of the bronze wire and not upon the earth's magnetic field. The result is that the zero is remarkably constant, often not changing by more than one-tenth millim. during several hours. The only disadvantage is that there is so little damping that, to bring the needle to rest in any reasonable time, it is necessary to use a coil of wire placed near the galvanometer in conjunction with a Leclanché cell and a tapping key. A little practice enables the observer to bring the spot quickly to rest.

The restoring couple varying as the angle of deflexion instead of as its sine, the time integral of a transient current is proportional to the angle of throw instead of the sine of half that angle. We verified by experiment that this law is accurately obeyed.

With rise of temperature the magnetic moment of a magnet diminishes, and we consequently found that with the same resistance in circuit the throw due to a given change of induction was rather less on a hot than on a cool day.

The logarithmic decrement depends upon the resistance in circuit with the galvanometer. But in every case the resistance of that circuit was kept constant during a set of observations, and thus all error due to this cause was avoided.

When we desired to draw a cyclic B—H curve for a specimen of iron, we practically followed the method described by Professor J. A. EWING*.

The Electro-dynamometer.

§ 27. The electro-dynamometer employed in the later experiments has a pair of fixed coils, each formed of 250 turns of No. 20 B.W.G. cotton-covered wire wound on ebonite bobbins. The mean radius of the coils is about 5 centims., and their resistance in series about 4 ohms. In order to avoid induced currents there were no large pieces of metal near the coils. In the central space hangs the suspended coil. This consists of 190 turns of No. 40 B.W.G. silk-covered wire. The mean radius of the coil is 1·7 centims. and its resistance about 28 ohms. The coil is attached along a diameter to a stiff brass wire, whose upper end carries a mirror. The mirror is placed so far above the centre of the coils that the beam of light from the lamp passes above the outside of the fixed coils. The moving coil is suspended by a phosphor-bronze wire $\frac{1}{1000}$ inch in diameter, and about 4 centims. long. The other connexion

* "Magnetic Induction in Iron and other Metals," 3rd Ed. Revd., § 192.

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with the moveable coil is formed by a second piece of the same bronze wire, about 6 centims. long, the total resistance of these two bronze wires being about 6 ohms. The time of vibration of the coil so suspended is about 9·5 seconds. It was found best to allow the lower bronze wire to be quite slack. The plane of the suspended coil then takes up a definite position due to the control of the bronze wires, and is unaffected by slight tiltings of the instrument. If the lower wire is pulled tight, and if the centre of gravity of the suspended system does not lie on the line joining the points of attachment of the bronze wire, a slight tilt of the instrument causes the coil to turn through a considerable angle.

The dynamometer was placed so that the axis of the fixed coils was at right angles to the magnetic meridian. The earth's magnetic force had in consequence no action upon the suspended coil in its equilibrium position. By diverting the primary current from the fixed coils and allowing the secondary current to flow through the suspended coil, we found that the earth's magnetic force produced no deflexion of the suspended coil. Our experiments are therefore free from any error arising from the action of the earth. The motion of the coil was observed with the aid of a lamp and scale in the ordinary way.

A little care is required in soldering the phosphor-bronze wire to its attachments. If the soldering bit is too hot, and if it comes into contact with the bronze, the solder alloys with the bronze to such an extent that the latter becomes very weak. Following a suggestion of Mr. W. G. Pye, we found that a strong joint is easily made in the following manner. The stout wire to which the fine bronze wire is to be soldered has a fine hole drilled along its axis; this hole is filled with solder which is kept melted by applying the soldering bit to the side of the wire. The fine bronze wire is then inserted into the melted solder, and the soldering bit is at once removed. The result is a very satisfactory joint.

The Reversing Keys.

§ 28. In the earliest experiments an ordinary (mercury) rocking key was used. Its defect is that, when contact is broken, the high resistance of the spark causes the current to sink to zero very rapidly. When contact is re-established the current rises at a rate depending upon the self-induction and the resistance of the circuit.

§ 29. To avoid all sparking, and to cause a very gradual change of the current, we next tried a liquid commutator similar to that used by Professor Ewing. A drum of insulating material, carrying two copper plates, A, B, revolves between two other plates, C, D, in a vessel containing a solution of copper sulphate. The plates C, D are connected to the battery, and the plates A, B to the primary circuit. The current flows in one direction in the circuit when A is close to C, and in the opposite direction when A is close to D. The reversal is thus very gradual. But the key was not altogether convenient, and its use was abandoned.

§ 30. In a third arrangement an ordinary (mercury) rocking key was used, a non-inductive coil D of 100 ohms being put in parallel with the primary circuit, PC, as in fig. 3. When the self-induction of the primary circuit is large the current does not stop very rapidly when contact is broken, for the primary current can still flow on through D, and thus the decay of the current is much less rapid than when it has to overcome the high resistance of the spark, as in § 28. When contact is re-established the current rises in exactly the same manner as with the simple rocking key. The practical objection to the method is that it does not allow of the current being directly measured by the convenient Weston ampere-meter, for the current must always flow in the same direction through this instrument. A Kelvin graded galvanometer or a shunted mirror galvanometer, MG, was accordingly used with this key.

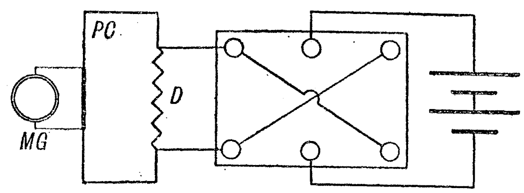

Fig. 3.

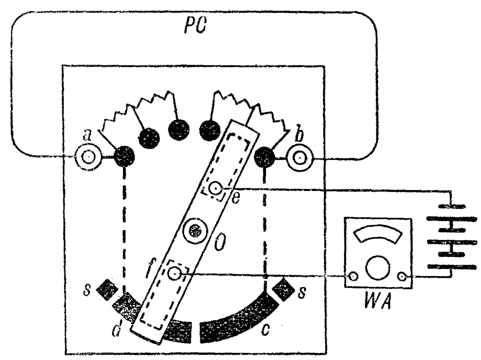

Fig. 4.

§ 31. The fourth reversing key used is shown in fig. 4. The battery is connected through the Weston ampere-meter WA with the terminals *e*, *f*, on an ebonite arm working about a pivot at O. The terminals *e* and *f* are connected to two brass springs, one at each end of the arm. One spring slides over a series of studs, while the other slides over two brass sectors *c*, *d*. Half the studs are connected to the terminal *a* by resistance coils in the manner shown in fig. 4, the remaining studs being connected to the terminal *b*. The sectors *d*, *c* are connected to the terminals *a*, *b*; the primary circuit, PC, joins the key at *a* and *b*. The resistance coils which connect the studs have resistances of 40 and 200 ohms. The two stops *s*, *s* serve to limit the motion of the arm, so that in its extreme positions the spring connected with *e* presses upon the stud nearest to either *a* or *b*. The width of the springs is sufficient to ensure that before the spring leaves one stud it is in contact with the next one.

When the arm is in either of its extreme positions the total resistance of the circuit is R + T, where T is the resistance of the battery and ampere-meter and R is the resistance of the rest of the circuit. When the spring connected with *e* comes on to the next stud the resistance is R + T + 40, and one more step makes the resistance R + T + 240. When the spring comes to the next stud the E.M.F. acting on the primary circuit is reversed.

Since the spring connected with e does not leave any stud before it touches the next one, as the arm is moved from one side to the other, the primary circuit is never broken except possibly in the central position of the arm. The circuit will be broken for an instant in the central position unless the spring connected with e touches the two central studs at the same instant that the spring connected with f touches both the sectors c, d. It is more or less a matter of chance whether this break of the circuit occurs, but if it does occur it is only after the current has been reduced to a small value by the introduction of the large resistance of 240 ohms. Except for this uncertainty we may say that the current is reversed in several steps which (except possibly those occurring in the uncertain part of the motion) are not sudden because of the great self-induction of the choking coil (§ 33). Though (with a possible exception) there is no sudden change in the current, the rate of variation of the current is no doubt much greater at some stages of the change than at others.

The key generally worked well; it was, however, subject to slight uncertainties.

§ 32. A fifth key was designed in 1900. Our aim was to ensure that the primary circuit should never be broken, and also that the resistances introduced into the circuit should be as small as possible, so that the rate of change of the current should, at every part of its variation, be as small as possible.

The battery is connected through the Weston ampere-meter, WA (fig. 5), with the terminals g, h on an ebonite arm working about a pivot at O. The terminals g, h are

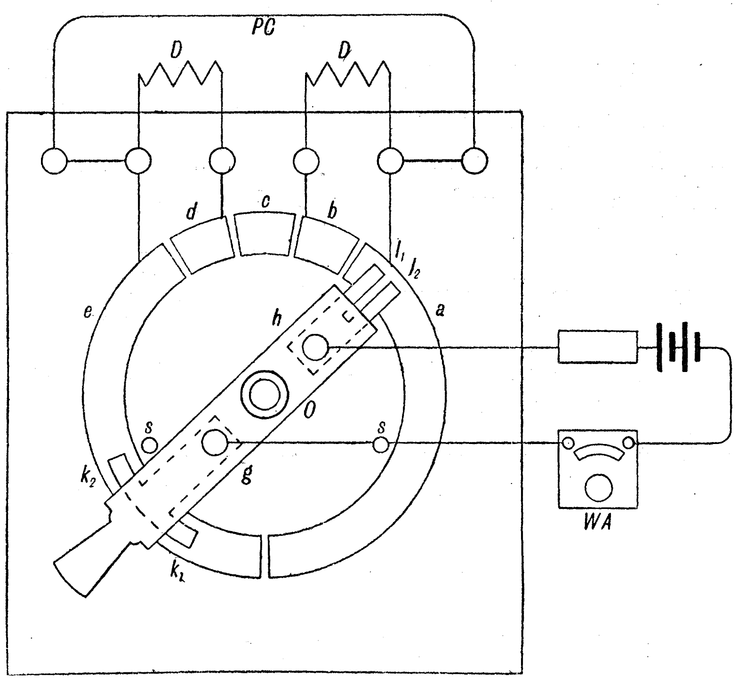

Fig. 5.

connected to two brass springs, k, l, one at each end of the arm. These springs slide over a series of sectors a, b, c, d, e, cut out of a brass ring. The sectors a and b and the sectors d and e are connected by the resistance coils D, D, while the sector c is insulated. The spaces between the sectors, about 1·5 millims. wide, are filled with ebonite so that the springs pass smoothly over them. The primary circuit PC joins the sectors a, e. The two stops s, s* arrest the motion of the arm when l has got well on to either a or e, so as to be clear of b or d. To avoid any possible break in passing from one sector to the next, the spring l is made in two portions, l_1, l_2, each of which presses upon the sectors. The spring k is arranged so as to touch the sectors only at its ends, k_1, k_2. The lengths of the sectors and of the springs are arranged so that the key performs the reversal in the following manner:—At the start l is on a, and k is on e; both parts of l reach b before k reaches a. While l is on b the primary current falls from E/(R + T) towards the limit E/(R + T + D), where E is the voltage of the battery, T the resistance of the battery and ampere-meter, and R the resistance of the rest of the circuit. Before l leaves b, k_1 reaches a, and then there is no E.M.F. acting round the primary circuit except what arises from the very small internal resistance of the key, and then also the resistance offered to any current flowing in the primary circuit is reduced from R + T + D to R. When l is on c, one pole of the battery is insulated, and there is no E.M.F. at all acting on the primary circuit, whose resistance remains R. When l reaches d, and k_2 is still on e, a very small E.M.F., due to the internal resistance of the key, acts in the reversed direction on the primary circuit. The resistance offered to a current in the primary circuit is still R. While l is on d and before it reaches e, k_2 leaves e; the resistance of the circuit is now raised to R + T + D, and the voltage E is introduced. When l reaches e the resistance is diminished to R + T, the voltage remaining E.

The resistances D need not be large; they are only used at all in order to save the battery and the ampere-meter. If the resistances D were made very small, and if T were small also, a large current would flow through the ampere-meter when l is on b, k_1 on a, and k_2 on e, causing damage to the meter. The resistances D need only be great enough to prevent the ampere-meter from being damaged by too strong a current.

We have only had opportunity to use this key in a few experiments, but as far as we have tested it we are satisfied with its action. The motion of the spot of light on the scale of the dynamometer is now quite free from the sudden jumps which it exhibited when the key of § 31 was used.

The Choking Coil.

§ 33. The "choking coil," which was inserted in the primary circuit to introduce great self-induction, has a core built up of 149 armature rings of an average thickness ·0714 centim., the inner and outer radii being 8·5 and 11·7 centims. respectively. The mean circumference is thus 63·7 centims. and the cross-section 34·0 sq. centims. If the core had been solid and not laminated, the eddy currents induced in it would

* By an oversight the wire from g to WA has been drawn through s.

have rendered the coil ineffective in "choking" any sudden variation of the current due to a sudden change of E.M.F. or of resistance. The iron, which was supplied by Messrs. CROMPTON, was tested for magnetic quality with the following results :—

H_0	·182	·371	·570	·854	1·09	1·53	2·10	3·37	4·98	6·90	8·58	11·15	16·20
B_0	35·4	103	190	390	660	1510	3090	5260	7300	8560	9470	10200	11500
μ	194	278	334	457	606	987	1470	1560	1470	1240	1100	914	710

In magnetic quality, the iron is very nearly the same as the soft thin sheet iron (Ring V.) tested by Professor EWING and Miss KLAASSEN.*

The core was wound with three independent layers of cotton-covered copper wire as follows :—

Layer.	B.W.G.	Turns.	Ohms.	Magnetic force per unit C.G.S. current.
1	No. 15.	225	·475	44·5
2	No. 15.	205	·50	40·4
3	No. 18.	265	1·44	52·3

Stout wire was used in order to avoid any considerable increase in the resistance of the primary circuit. This resistance must be kept low if the choking action is to be efficient.

§ 34. This choking coil is perhaps unnecessarily large. It might be better to use a core smaller both in diameter and cross-section and to form each of the coils on it with wire of the same gauge as that used for a corresponding coil on the solenoid. By this expedient, if we always employ corresponding coils on the choking coil and the solenoid, the magnetic force in the choking coil has the same value as in the solenoid. Thus, if the iron used in the choking coil is of approximately the same quality as the specimen under test, the coil will most effectively choke the primary current approximately at the time when the choking is most needed, viz., when the value of dB/dH for the specimen has its greatest value, as was explained in § 10. If the iron plates used in the core of the choking coil are of good quality it is very unlikely that any specimen will be so "soft" in the magnetic sense as to have a value of dB/dH so many times greater than the value of dB/dH for the core as to lead to inaccurate measurements. If the specimen is of "hard" iron or steel the value of dB/dH for the specimen is by comparison so small that it is of little consequence if the most efficient choking action takes place when dB/dH for the specimen is not at its

* "Magnetic Qualities of Iron," 'Phil. Trans.,' A, vol. 184 (1893), p. 1003.

maximum value. It would be necessary to make the core smaller than in the coil used by us, so that in spite of the smaller section of the wire the resistance might still be small.

Arrangement of the Apparatus.

§ 35. After this description of the apparatus we proceed to explain, by the help of fig. 6, how it was arranged. The battery SC, consisting of one or more small storage cells, was connected through the adjustable resistance R_1 and the Weston ampere-

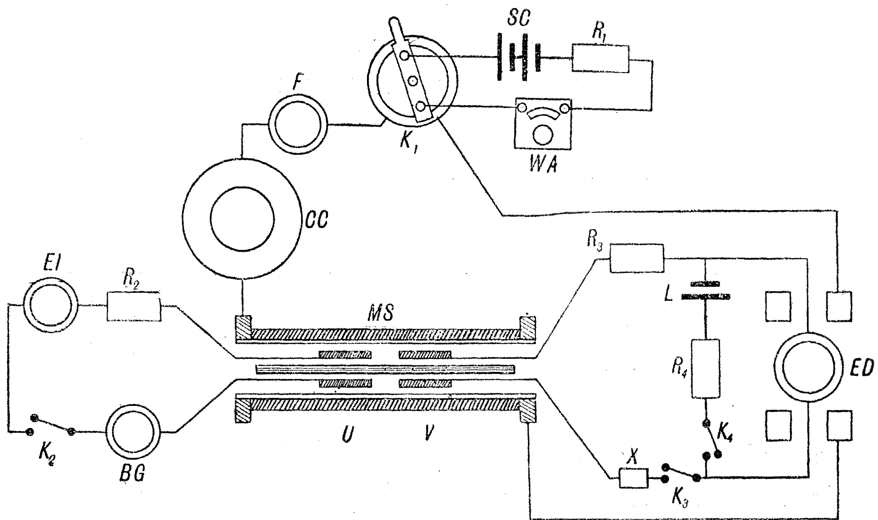

Fig. 6.

meter WA to the special reversing key K_1. An adjustable length of German-silver wire included in the circuit enabled us to keep the current to a definite value for any length of time. On leaving the key K_1, the current passes round the fixed coils of the electro-dynamometer ED, round the magnetising solenoid MS and the choking coil CC, and finally round a compensating coil F. By adjusting the position of F, the small effect upon the ballistic galvanometer of the current passing through ED and MS was completely annulled.

The circuit of the ballistic galvanometer BG contains a resistance box R_2 and the earth-inductor EI, as well as the secondary coil U and the key K_2.

The circuit of the suspended coil contains the resistance box R_3, the secondary coil V, the key K_3, and the resistance coil X formed of copper wire adjusted to have the same resistance as the earth-inductor EI, as described in § 23.

To bring the suspended coil to rest a current of the order of 1/200,000 ampere was sent through the suspended coil by depressing the tapping key K_4, a Leclanché cell L providing the current. In the actual experiments a system of shunts was used instead of the single high resistance R_4, which, for the sake of simplicity, is represented in the diagram; the effect is, however, the same in both cases. When we desired to damp the more violent motions of the coil, R_4 was diminished so that a current of 1/20,000 ampere could be sent through the suspended coil, the key K_3 being opened to prevent any of this stronger current from passing round the secondary coil and affecting the magnetisation of the specimen. The final damping was always done with K_3 closed and R_4 at its large value. We verified by experiment that currents many times larger than those actually employed for this purpose had no appreciable effect upon the hysteresis of the specimen.

When the dynamometer is to be standardised the resistance coil X is taken out of the circuit and the earth-inductor is put in its place.

Practical Example of the Method.

§ 36. To illustrate the working of the method, we now give a set of observations made to determine the hysteresis loss, W, as well as the mean maximum induction B_0, for a definite range of magnetic force $\pm H_0$, for a specimen of soft iron :—

Area of cross-section (circular)	$A = \cdot 178$ sq. centim.
Change of induction due to earth inductor	$P = 87200$.

Magnetic force per unit C.G.S. current .	$4\pi N = 212 \cdot 2$.
Maximum current	$C_0 = \cdot 0706$ C.G.S.
Maximum magnetic force	$H_0 = 15$.

Number of turns on secondary coil . . .	$m = 25$.
Throw of galvanometer due to earth inductor	$\delta = 12 \cdot 73$ centims.
Mean throw on reversing primary current	$\beta = 15 \cdot 63$ centims.

Mean maximum induction $B_0 = P\beta/2mA\delta = 770\beta = 12030$.

Current when earth inductor is turned .	$C' = \cdot 0527$ C.G.S.
Throw of dynamometer due to earth inductor	$\phi = 4 \cdot 68$ centims.
Number of turns on secondary coil . . .	$n = 300$.
Resistance of secondary circuit	$S = 43$ ohms.
Length of secondary coil	$l = 6 \cdot 6$ centims.
Mean area of one turn of secondary coil .	$G = \cdot 65$ sq. centim. $= 3 \cdot 66A$.
Throw, H from $+15$ to -15 ($E = 8$ volts)	$\theta_1 = 23 \cdot 85$ centims.
Throw, H from -15 to $+15$ ($E = 8$ volts)	$\theta_2 = 23 \cdot 30$ centims.

Hence $U = \frac{C'PN}{An\phi}(\theta_1 + \theta_2) = 311 \cdot 8\,(\theta_1 + \theta_2) = 14700$ ergs per cub. centim. per cycle.

In order to find Z, the observations for θ_1 and θ_2 were repeated with other voltages (§ 15), with the result

$$E = 8 \qquad E' = 12 \qquad E'' = 16 \text{ volts.}$$
$$U = 14700 \qquad U' = 15130 \qquad U'' = 15540$$

From U and U′, $Z = (U' - U)E/(E' - E) = 860$

From U and U″, $Z = (U'' - U)E/(E'' - E) = 840$

Mean value of $Z = 850$ ergs per cub. centim. per cycle.

Hence $W = U - Z = 14700 - 850 = 13850$ ergs per cub. centim. per cycle.

In this case L and M are negligible, and thus, as in § 15, $X/Y = QSl/n^2\sigma$. Taking σ as 10^{-5} ohm per centimetre cube, and putting $Q = 1/8\pi$, since the section is circular, we have $X/Y = 12{\cdot}5$. But $Z = X + Y = 850$, and hence

$$X = 787, \quad Y = 63 \text{ ergs per cub. centim. per cycle.}$$

The value of Y is so small in comparison with U that it could not be determined satisfactorily by varying S (§ 14). A series of experiments was made in which S was varied, but the small irregularities rendered the observations useless for the determination of so small a quantity as Y is in this case. [See § 41 (*d*).]

§ 37. A word should perhaps be said as to the accuracy aimed at in our experiments. The throws were all recorded to $\frac{1}{10}$ millim., the throws themselves varying from 25 centims. to 1 centim. The difference between $\frac{1}{2}\theta$ and $\frac{1}{4}\tan 2\theta$, seldom amounting to more than 1 per cent., and usually much less, has generally been neglected.

The calculations were mostly effected by a 10-inch slide-rule, and the numbers recorded in the tables are the numbers read from the rule.

Tests of the Accuracy of the Method.

§ 38. We may now pass on from the theory of the method, and the description of the apparatus used for applying it in actual measurements of hysteresis, to an account of the tests which we have made in order to determine if, in our experiments, the theoretical conditions are so nearly satisfied that the dynamometer yields accurate measurements of hysteresis.

§ 39. *Test by Comparison with the Cyclic B—H Curve.*—We were without the guidance of the completed theory till 1900, and thus till that date we did not know what measurements were required for the determination of the correction arising from the eddy currents and the finite conductivity of the secondary circuit. Under these conditions the only way of testing the accuracy of the dynamometer method was to make a cyclic B—H curve by the use of the ballistic galvanometer, to calculate W from the area of the curve, and to compare this value with the value of U (§ 13) found by the dynamometer for the same range of magnetic force. We made this test on many occasions with different specimens and with various keys. We now give the results in the following table, where the third and fourth columns give respectively the value of W found by the ballistic galvanometer and the value of U found by the dynamometer; in the last column the section of the specimen is given. The numbers in this column, such as (3), refer to the first column.

No.	Date.	$W_{(BG)}$.	$U_{(ED)}$.	H_0.	B_0.	*No Choking Coil used.*
1	7 Nov., 1895	7304	16460	7·98	10340	Iron wire, ·0201 sq. centim. Ordinary key, § 28.
2	9 Nov., 1895	7409	17780	8·34	10170	Iron wire (1). Ordinary key.
3	11 Dec., 1895	11500	11620	10·48	8050	Ring formed of iron wire, total section ·8527 sq. centim. Ordinary key.
4	16 Dec., 1895	6850	6710	7·65	5790	Ring (3). Key of § 29.
5	16 Dec., 1895	11570	12480	9·74	14370	Iron wire, ·00697 sq. centim. Key of § 29.
6	24 Mar., 1896	6160	6288	7·10	7500	Ring of iron wire, ·7611 sq. centim. Key of § 30.
						Choking Coil used.
7	25 Mar., 1896	3470	3616	4·87	5480	Ring (6). Key of § 30.
8	25 Mar., 1896	1013	1013	3·00	2460	Ring (6). Key of § 30.
9	27 July, 1897	2827	2946	3·40	6050	Core of iron rings, 3·622 sq. centims.; thickness of each ring, ·207 centim.; radii, 3·83, 6·02 centims. Ordinary key.
10	12 July, 1898	5260	5408	4·97	8200	Iron wire, ·00708 sq. centim. Ordinary key.
11	14 July, 1898	11905	11960	9·78	13600	Iron wire (10). Ordinary key.
12	19 July, 1898	241·6	330	7·46	950	Six steel wires; total area, ·02472 sq. centim. Ordinary key.
13	16 Aug., 1898	1720	1780	8·09	1980	Mild steel rod, ·0938 sq. centim. Key of § 30.
14	17 Aug., 1898	633	726	8·09	1200	Steel rod (13) under torsion. Key of § 30.
15	7 Aug., 1900	74250	75200	35·71	16460	Ten pianoforte steel wires; total cross-section, ·0255 sq. centim. Key of § 30.
16	8 Aug., 1900	16440	17480	35·71	15550	Ten soft iron wires, hardened by stretching; total cross-section, ·0412 sq. centim. Key of § 30.
17	16 Aug., 1900	7098	7400 (*a*) 7275 (*b*)	10·65	9570	Iron wires (16). Key of § 32. (*a*) Before, (*b*) after observations for B—H curve. Compare § 46.

§ 40. It will be seen that there is very fair agreement between the values found for W and U, except in the cases of (1) and (2), when there was no choking coil in the circuit, and only an ordinary mercury key was used. Except in (12) and (14) the agreement is perhaps as close as could be expected, when we consider how much the behaviour of iron, specially under small magnetic forces, depends upon its previous history. In making a cyclic B—H curve, if we change H from H_0 to H_1, and find that B changes from B_0 to B_1, then B_1 is the value assumed by B after H_1 has acted for a finite time—about one-quarter of the period of vibration of the galvanometer needle. But in the dynamometer method H does not halt for any appreciable time at any intermediate value as it changes from H_0 to $-H_0$, and thus the value of B corresponding to H_1 may differ somewhat from the value B_1 found by the galvanometer.

The experiments of §§ 45–49 below show that, for a given value of H_0, W may vary considerably from causes depending upon the magnetic history of the iron even when its temperature and strain remain constant. Thus, unless it is possible to give an accurate account of the history of the iron and of the manner in which the magnetic force changes from H_0 to $-H_0$, it is impossible to assign any definite meaning to W, and hence, in the absence of such an account, it is useless to attempt a very close comparison between the value of U found by the dynamometer and the value of W found from the cyclic B—H curve. But the agreement between the two values as recorded in the table will perhaps suffice to give general confidence in the method.

§ 41. *Test by Variation of the Resistance of the Secondary Circuit.*—The process of comparing the value of U found by the dynamometer with the value of W found from the area of the cyclic B—H curve is laborious, and after all does not furnish an absolute test of the equality of U and W for that particular law of variation of the magnetic force which is obeyed when the measurement is made, as was explained in § 40. When, as was the case in most of our experiments, the specimen is a fine wire not exceeding 2 millims. in diameter there is a second method, described in § 14—it may be termed, in contrast with the first, a self-contained method—which is easy of application, enabling us to test the accuracy of the dynamometer measurements by the dynamometer itself. If, in any case, we can find the effect of Y, we have a superior limit to the effect of X, for by (14) $X/Y < QSl/n^2\sigma$. We now give the results of some applications of this second method.

It will generally suffice to take two values of S, one double the other. Since, when S is varied, ϕ is inversely proportional to S, it follows that U $[= NC'P(\theta_1 + \theta_2)/An\phi]$ is proportional to $S(\theta_1 + \theta_2)$, provided that ϕ and θ are corrected for damping (§ 14). Hence, in testing the accuracy of the dynamometer measurements, it is sufficient to compare the values of $S(\theta_1 + \theta_2)$ for the two values of S. If S be doubled the consequent diminution in $S(\theta_1 + \theta_2)$ is, by § 14, equal to the amount which must be subtracted from the product for the larger resistance in order to obtain the value of the product corresponding to $S = \infty$.

(*a.*) We will first refer to the results obtained by one of us in November, 1895, for an iron wire ·0201 sq. centim. in section, with $H_0 = 8{\cdot}34$ and $B_0 = 10170$, the value of U being 17780 when S = 59·7 ohms, while W was found from a cyclic B—H curve to be 7409. A simple mercury rocking key was used, and the primary circuit had only the self-induction of the dynamometer and the solenoid. As S rose from 59·7 to 584 ohms, U fell from 17780 to 14550, being closely represented by $U = 14070 + 221900/S$. From this we find by (20) that when S = 59·7 ohms, Y = 3720. In these experiments $n = 1285$, $l = 11{\cdot}5$ centims., G = ·785 sq. centim., and hence if L and M (§ 11) be neglected, $X/Y = 1{\cdot}66$ when S = 59·7 ohms. Thus $Z = X + Y = (1{\cdot}66 + 1)3720 = 9890$, when S is 59·7 ohms. Subtracting this from the corresponding value of U, viz., 17780, we obtain W = 7890. Thus, when the proper corrections are applied, the method yields approximately correct results even with an ordinary key and without a choking coil. It was a little disheartening to find so great a discrepancy between W and U as the numbers 7409 and 17800 indicated, and we were glad to find five years later that the discrepancy is satisfactorily accounted for by the more complete theory.

(*b.*) Using the key of § 31 as well as the choking coil, we found (August 8, 1899) for a soft iron wire ·0324 sq. centim. in section, with $H_0 = 18{\cdot}75$, $B_0 = 15800$, W = 19400, after correction for damping and for finite arcs,

$$S = 59{\cdot}5 \text{ ohms}, \quad \theta_1 + \theta_2 = 34{\cdot}01 \text{ centims.}, \quad S(\theta_1 + \theta_2) = 2024,$$
$$S = 118{\cdot}5 \text{ ohms}, \quad \theta_1 + \theta_2 = 17{\cdot}08 \text{ centims.}, \quad S(\theta_1 + \theta_2) = 2025.$$

For the secondary coil $n = 1285$, $l = 11{\cdot}5$; also $Q = 1/8\pi$, $\sigma = 10^{-5}$, and thus, when S = 59·5 ohms, $X/Y < QSl/n^2\sigma < 1{\cdot}66$. But the effect of Y, the correction arising from the conductivity of the secondary circuit, in causing a change in the product $S(\theta_1 + \theta_2)$, is quite insensible, and hence the eddy current effect, X, is negligible. In this case, as closely as we could measure, U = W.

(*c.*) With the same key and choking coil we found (August 7, 1900) for a bundle of ten pianoforte-steel wires, of total section ·0255 sq. centim., with $H_0 = 35{\cdot}7$, $B_0 = 16460$, W = 75200,

$$S = 39{\cdot}2 \text{ ohms}, \quad \theta_1 + \theta_2 = 22{\cdot}16 \text{ centims.}, \quad S(\theta_1 + \theta_2) = 869,$$
$$S = 78{\cdot}8 \text{ ohms}, \quad \theta_1 + \theta_2 = 11{\cdot}04 \text{ centims.}, \quad S(\theta_1 + \theta_2) = 870.$$

Here $n = 600$, $l = 9{\cdot}6$, so that when S = 39·2 ohms $X/Y < QSl/10n^2\sigma < {\cdot}416$. Here we have divided by 10, since the specimen is formed of ten wires (§ 9). In this case X and Y are both negligible.

(*d.*) To illustrate the method of finding the correction by doubling the resistance, we take a test made on a bundle of ten iron wires coated with shellac varnish to prevent eddy currents from wire to wire. The total section was ·0412 sq. centim. With the key of § 31, but with an inefficient choking coil, we found (August 9, 1900), when H = 35·8, $B_0 = 15550$,

$$S = 58{\cdot}6 \text{ ohms}, \quad \theta_1 + \theta_2 = 6{\cdot}68 \text{ centims.}, \quad S(\theta_1 + \theta_2) = 391,$$
$$S = 118{\cdot}0 \text{ ohms}, \quad \theta_1 + \theta_2 = 3{\cdot}21 \text{ centims.}, \quad S(\theta_1 + \theta_2) = 379.$$

The difference between the products is 12, and thus the correction to be subtracted from 391 is 24. We had $n = 1285$, $l = 11{\cdot}5$ centims., so that since there are ten wires, when S = 58·6 ohms, $X/Y < {\cdot}162$. But we make only a small error in taking X/Y = ·162, and thus the total correction to be subtracted from 391 is 24 × 1·162 or 28. Hence the value of $S(\theta_1 + \theta_2)$ to be used in finding U is 363.

It was only in the later experiments that we were guided by the complete theory to test the accuracy of the measurements by varying S, but from the tests just described we may conclude that the value of U, obtained from the dynamometer throw, was in all our experiments very nearly equal to W.

Energy dissipated by Eddy Currents.

§ 42. In our experiments the ratio X/Y was small, so that, Y being small in comparison with U, X was small also, and thus U and W were nearly equal.

We now consider the case in which the section of the specimen is large, so that S has to be made large and n small, in order to reduce the sensitiveness of the apparatus. Under these conditions X/Y is large, while Y is now so small in comparison with U that it eludes observation, and thus cannot be determined by varying S, the resistance of the secondary circuit. These conditions have prevailed in the experiments made by Mr. R. L. WILLS with some of our apparatus; in these experiments the eddy current loss was so large that systematic measurements were made to determine X for every value of H_0 employed.

Mr. WILLS used the key described in § 32, and found that for a given total resistance of the primary circuit the "throw" of the dynamometer is practically independent of the resistances denoted by D in the description of the key. This key divides the primary circuit into two portions; the resistance of the portion which includes the battery is denoted by T, and the resistance of the other portion by R. When the voltage E was to be changed in order to make the observations suggested in § 15, Mr. WILLS changed the total resistance, R + T, of the primary circuit, mainly by changing R. The resistance T was small compared with R, and was used as a means of obtaining an exact adjustment of the current to definite values. No attempt was made to make T bear any fixed ratio to R.

The reversal of the current does not take place quite in the manner described in § 15. In addition to the effect of the resistances D, there is the further point of difference that although R + T is adjusted to be accurately proportional to E, the voltage driving the current C_0, yet R is not accurately proportional to E because T does not bear any fixed ratio to R. The value of dC/dt and, consequently, the power absorbed by the eddy currents while the primary current is sinking to zero is thus not quite proportional to E for a given value of C.

During the subsequent rise of the current the whole resistance R + T comes into play, and, the resistances D being comparatively small, the value of dC/dt for a given value of C is nearly proportional to E.

Now the increment of current dC contributes to X a quantity proportional to $(dB/dH)^2 . dC/dt$. Thus although, for a given value of C, dC/dt is much greater during the fall than during the rise of the current, because dB/dH for the iron of the choking coil is much smaller during the first than during the second stage, yet, for specimens of high permeability, $(dB/dH)^2$ is enormously greater during the second than during the first stage. Thus, for a given value of C, the product $(dB/dH)^2 . dC/dt$ during the first stage is small compared with its value during the second stage. Hence by far the greater part of the eddy current loss occurs during the rise of the current. In fact, if the choking coil have a core of the same iron as the specimen and be similarly wound, and if the current be reversed in the manner described in § 15, the eddy current losses during the fall and subsequent rise of the current are proportional to R × area *adb* and (R + T) × area *ac'b'e* respectively, the areas being shown in fig. 1. Thus the main part of the eddy current loss occurs during the rise of the current; this part is very nearly proportional to E. The eddy current loss during the fall of the current is indeed only roughly proportional to E, but it is small in comparison with the loss during the rise of the current. Hence the total eddy current loss during a semi-cycle is nearly proportional to E, and thus can be determined approximately by the formula $X = (U' - U) E/(E' - E)$, as explained in § 15, since Y is now negligible, and X is thus sensibly equal to Z.

As illustrations we now give two examples kindly furnished us by Mr. R. L. WILLS. In the first example the specimen was a portion, 2·49 centims. in length, of a circular tube of radii 3·810 and 3·185 centims. A plane through the axis forms a rectangular section of the ring, the sides of the rectangle being $a = 2{\cdot}49$, $b = {\cdot}625$ centim. Thus $A = 1{\cdot}56$ sq centims., $a/b = 4$, while the mean circumference is $l = 22$ centims. Hence, treating the ring as a straight rod, we find by Appendix I., $Q = {\cdot}0176$. Now the resistance, S, of the secondary circuit varied from 23 to 523 ohms, while n, the number of turns, was 50, so that X/Y $[= QSl/n^2\sigma]$ varied from 356 to 8300. Hence Y was negligible in comparison with X, and X was thus sensibly equal to Z.

For each value of H_0 three determinations of U were made with batteries of 8, 16, and 24 volts. The value of W was deduced from the three values of U by the formulæ $W = 2U - U'$, $W = 3U' - 2U''$, which follow from § 15. In the last column we give the value of X, the space-average of the eddy current loss corresponding to E = 8 volts, X as well as W being expressed in ergs per cub. centim. per cycle.

H_0.	B_0.	S. ohms.	U. E = 8.	U'. E' = 16.	U''. E'' = 24.	2U − U'.	3U' − 2U''.	W. (mean).	X. E = 8.
·34	194	23	5·9	7·3	8·8	4·5	4·3	4·4	1·5
·68	540	23	32·3	38·2	44·1	26·4	26·4	26·4	5·9
1·02	1205	23	154	169	185	139	137	138	16
1·36	2657	73	585	655·4	721	514·6	524·2	519·4	65·6
1·70	4629	123	1528	1640	1736	1416	1448	1432	96
2·04	6107	223	2471	2702	2919	2240	2268	2254	217
2·38	7139	223	3396	3714	4017	3078	3108	3093	303
2·72	8077	523	4242	4683	5124	3801	3801	3801	441
3·06	8749	523	5159	5702	6211	4616	4684	4650	509
3·40	9332	523	5940	6551	7161	5329	5331	5330	610
4·02	10150	523	7297	8146	8960	6448	6518	6483	814

The agreement between 2U − U′ and 3U′ − 2U″ is satisfactory; it may be taken as evidence that the theory sketched in § 15 is practically applicable to Mr. WILLS's experiments. The ratio W/Y ranged from 1040 to 66,000, and thus no appreciable error was introduced by neglecting the correction due to the conductivity of the secondary circuit.

In the second example the specimen was a ring similar to the last, the numbers now being $a = 1·985$, $b = ·735$, $l = 20$ centims., A = 1·46 sq. centims., $b/a = ·37$, Q = ·0236, $n = 50$. The material was an alloy of iron and aluminium, and the experiments were made when the specimen was at 645° C. We do not know σ; if it was 10^{-5}, then X/Y varied from 430 to 1380. The results are shown in the following table, where it will be seen that there is again good agreement between 2U − U′ and 3U′ − 2U″. The agreement is the more significant because the eddy current loss, X, now forms a very considerable part of the whole energy dissipated in each cycle.

H_0.	B_0.	S. ohms.	U. E = 8.	U′. E′ = 16.	U″. E″ = 24.	2U − U′.	3U′ − 2U″.	W. (mean).	X. E = 8.
·20	1747	23	77·2	80·3	83·4	74·1	74·1	74·1	3·1
·34	3357	23	223·8	251·5	276	196·1	202·5	199·3	24·5
·48	4406	23	332·0	390·8	448	273·2	276·4	274·8	57·2
·68	5507	73	461	570	669	352	372	362	99
1·02	6661	73	615	773	926	457	467	462	153
1·36	7238	73	714	917	1107	511	537	524	190
2·04	8282	73	858	1096	1335	620	618	619	239
2·72	8654	73	952	1214	1472	690	698	694	258
3·40	8968	73	1007	1264	1528	750	736	743	264

Complete Cycles and Semi-cycles.

§ 43. The theoretical investigation of § 8 shows that the throw due to a complete cycle should be equal to the sum of the throws due to a pair of semi-cycles. We have not spent any considerable time in testing this point, for the arrangements used by us have not been well adapted for that purpose. In order that U shall be sensibly equal to W, it is necessary, when the specimen is a thin wire of soft iron, that the self-induction of the choking coil should be very large, and in this case, after the current has been reversed by the key, it does not at once rise to its full strength. Thus, if the key be worked very rapidly so as to make a pair of semi-cycles in quick succession, the current may never rise to its full value in the middle of the cycle. On the other hand, if a definite halt be made after the current has been reversed for the first semi-cycle, the dynamometer coil will have moved considerably from its zero position when it receives the impulse due to the second semi-cycle. It is perhaps for these reasons that we have not found good agreement between the throw due to a complete cycle and the sum of the throws due to a pair of semi-cycles. We have not taken any special steps to secure this agreement because (1) we are satisfied that the sum of the throws for a pair of semi-cycles gives a correct measure of the hysteresis, and (2) it was more convenient to take a pair of semi-cycles than a complete cycle, since with the semi-cycles we could measure B_0 and W simultaneously.

We found that the throw for a complete cycle depended a good deal upon the way in which the key was manipulated. On the other hand, the throws for each of a pair of semi-cycles were generally very regular.

§ 44. In § 6 we show that, when, on account of the previous magnetic history of the specimen, the two throws of the dynamometer for a pair of semi-cycles are unequal, the sum of the throws still furnishes a correct measure of the energy dissipated in the complete cycle, if U = W. This inequality in the two throws gave us much trouble. It persisted in some cases in spite of very many reversals of the magnetic force. As tested by the ballistic galvanometer, the iron had reached a steady state, but the galvanometer only shows the change of induction on reversing H_0, and not the actual values of B corresponding to $+ H_0$ or to $- H_0$. In the belief that after many reversals the cyclic B—H curve would lie symmetrically about the axes of B and H, we naturally did not look to the iron for the cause of the inequality in the throws. We spent some weeks in making changes in the dynamometer and other parts of the apparatus, but of course without result. At times the inequality would almost disappear (probably on account of a large number of reversals), only to reappear for some apparently slight cause, such as changing the range of the magnetic force, or using another specimen, and this uncertainty made the matter very perplexing. But though we felt that we had failed to discover the cause of the inequality, yet the agreement between the value of U, found by the dynamometer, with that of W, deduced from the cyclic B—H curve, made us certain that the method was giving at least approximate results, and we began to make experiments on the effects of tension and torsion upon hysteresis. During these experiments we found that the inequality varied as the stress was applied, and we were thus led to see that the origin of the inequality lay in the iron. When once we suspected the cause, it was easy to make experiments to satisfy ourselves that our suspicion was true. We were then able to employ the method without hesitation.

§ 45. To study the inequality in the two throws for a pair of semi-cycles in a definite manner we made the following experiments :—A freshly annealed iron wire, which had not been magnetised since the annealing, was placed in the solenoid when no current was flowing, and the magnetic force H_0 was then applied for the first time. We found that the throw of the dynamometer for the first reversal from H_0 to $-H_0$ was greater than for any subsequent reversal in either direction, and that up to 100 reversals the throw is greater when the magnetic force changes from H_0 to $-H_0$ than when it changes from $-H_0$ to H_0. In these experiments the value of H_0 was 5·77, and the value of B_0, after 100 reversals, 9990; the section of the wire was ·0265 sq. centim. An ordinary mercury rocking key was used, and the choking coil was inserted in the circuit.

We now give the results of one of these experiments. The numbers heading the columns indicate the cycle to which the pair of semi-cycles in any column belong.

Cycle.	1.	2.	3.	4.	5.	6.	28.	50.
$\frac{1}{4\pi}\int HdB$ for $+H_0$ to $-H_0$	5440	4820	4650	4570	4480	4470	4260	4190
$\frac{1}{4\pi}\int HdB$ for $-H_0$ to $+H_0$	4540	4380	4290	4230	4210	4160	4030	3990

In this example the inequality is greater and more persistent than in most cases. The inequality is generally much more marked for small than for large magnetic forces. It will be noticed that the hysteresis diminishes with continued reversals.*

Effect of Continued Reversals.

§ 46. In the course of the experiments undertaken in the hope of finding the cause of the inequality of the two throws of the dynamometer, θ_1 and θ_2, for the two semi-cycles belonging to a single cycle, we had occasion to put the specimen through many cycles. We then discovered that the hysteresis diminishes very considerably with continued reversals of the magnetic force. To investigate this matter more completely a systematic set of experiments was made in 1899 with the object of determining how the effect depends upon the limits of the magnetic force. The experiments were carried out in the following manner:—The specimen, demagnetised by annealing or "by reversals," was placed in the solenoid when no current was flowing. The magnetic force was first applied in the positive direction; the magnetic force was then reversed from H_0 to $-H_0$ and the throws of the dynamometer and the galvanometer were read simultaneously. The next reversal $-H_0$ to H_0 was observed in like manner. These two reversals constituted the first cycle. The observations were repeated for other cycles as shown in the tables, where in each case the first column shows the number of the cycle. The two throws of the dynamometer, θ_1 and θ_2, sometimes showed an inequality which was rather persistent, though it was never greater than that recorded in § 45. We have therefore thought it sufficient to give the value of the hysteresis deduced from each complete cycle. The mean maximum magnetic induction, B_0, which diminished in much the same manner as the hysteresis, is also recorded in the tables.

* The initial asymmetry of the hysteresis loop and its gradual shrinking with continued reversals are well shown in fig. 154 in Professor EWING's 'Magnetic Induction . . .', 3rd Edition. Much labour would have been saved had we realised the significance of Professor EWING's curves.

The experiments of Professor J. A. EWING* on soft iron show that after the few dozens of reversals which are necessary to avoid the effects of the initial diminution of B_0, the values of B_0 and of W remain constant, even after 70,000,000 of cycles of reversal when the temperature of the iron is not allowed to rise above that of the atmosphere. In his experiments the maximum magnetic force H_0 in these cycles was about 4, and the maximum induction, B_0, about 8000. It is now well known that if iron be raised to a temperature only 50° C. above the atmospheric temperature, and be maintained in that state for many hours, a large increase in the hysteresis often results.

§ 47. *Soft Iron demagnetised by Annealing.*—The specimen was a wire of soft iron ·0265 sq. centim. in section. In each case it was annealed before the observations for any given value of H_0 were made; the previous magnetic history of the iron was thus wiped out. The key of § 31 was used.

No.	$H_0 = 2·50$.		$H_0 = 4·98$.		$H_0 = 7·57$.		$H_0 = 10·86$.		$H_0 = 14·70$.	
	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.
1	2220	598	8150	5250	10550	9500	11800	12450	12840	14610
2	2040	519	7970	4870	10430	8980	11680	12050	12830	14340
3	2000	488	7950	4730	10340	8830	11660	11960	12760	14230
11	1940	452	7740	4530	10250	8520	11660	11850	12730	14140
21	1890	440	7700	4420	10250	8590	11670	11760	12720	13990
41	1840	433	7650	4360	10220	8450	11660	11760	12730	14070

No.	$H_0 = 20·05$.		$H_0 = 25·15$.		$H_0 = 30·24$.		$H_0 = 40·65$.	
	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.
1	13480	17420	13270	20020	14360	21300	14760	22600
2	13430	17300	13260	19850	14300	21260	14740	22600
3	13420	17200	13280	19770	14300	20940	14720	22610
11	13420	17060	13230	19690	14290	20820	14720	22400
21	13430	17100	13280	19720	14290	20900	14730	22560
41	13420	17080	13250	19640	14310	20820	14710	22630

It will be seen that the effect of continued reversals of H_0 in producing a diminution of B_0 and W rapidly decreases as H_0 increases, both B_0 and W becoming sensibly constant when H_0 reaches the value 40·65. The following table illustrates this fact, and shows that the percentage change of W is always greater than the percentage change of B_0. The change recorded is that which occurred in the first 41 cycles :—

* 'The Electrician,' vol. 34, January 11, 1895.

H_0.	2·50	4·98	7·57	10·86	14·70	20·05
Percentage change in B_0 .	17·1	11·0	3·1	2·0	·9	·4
Percentage change in W .	27·5	17·0	11·0	5·5	3·7	1·9

§ 48. *Soft Iron demagnetised "by Reversals."**—The same piece of wire and the same key were used as in § 47. The wire was very completely demagnetised by reversals before the tests for B_0 and W were made; a mirror magnetometer served as indicator. The results are shown in the table below:—

No.	$H_0 = 2·50$.		$H_0 = 7·45$.		$H_0 = 20·50$.	
	B_0.	W.	B_0.	W.	B_0.	W.
1	2068	711	9950	8000	13480	16670
2	1997	662	9980	8000		
3	1984	669	9960	8000		
4	1997	652				
11	1960	638	9940	7910	13510	16590
21	1960	633	9940	7850		
41	1950	613	9890	8000	13510	16580
81	1913	603			13480	16610
101			9900	7820		

For $H_0 = 7·45$ and $H_0 = 20·50$ the values of B_0 and W are constant within the errors of observation. For $H_0 = 2·5$ there is a considerable diminution in both B_0 and W during the first 41 cycles, but much less than in the case of the wire demagnetised by annealing, the changes in the first 41 cycles being now for B_0 5·8 per cent., and for W 13·8 per cent.

The wire had been annealed before the tests for $H = 2·5$ in § 47 were made. Without further annealing, the tests of the present section for $H_0 = 7·45$, 20·50, and 2·5 were made in the order written. This treatment has had the result of diminishing the initial value of B_0 and raising its value after 41 reversals; the value of W is at the same time considerably increased.

§ 49. *Steel Rod.*—A few experiments were made upon a steel rod ·0925 sq. centim. in section, using the key of § 31. We endeavoured to demagnetise it by reversals before the tests for B_0 and W were made, but we were only partially successful. The dynamometer was not sensitive enough to enable us to go to small magnetising forces. The induction remained practically constant for both values of H_0; in 41 cycles the hysteresis fell by 10 per cent. when $H_0 = 7·29$ and by 2·5 per cent. when

* See EWING, 'Magnetic Induction . . .," 3rd Edition, fig. 155.

$H = 20 \cdot 54$, the changes in W being roughly the same as those for the iron wire, demagnetised by annealing, under approximately the same magnetic forces.

No.	$H_0 = 7 \cdot 29$.		$H_0 = 20 \cdot 54$.	
	B_0.	W.	B_0.	W.
1	1275	853	11430	32840
2	1255	838	11370	32670
3	1275	838	11370	32530
4			11370	32580
11	1240	787	11310	32110
21	1275	779	11340	32110
41	1275	768	11280	32040

$W - B_0$ Curves for Zero Stress.

§ 50. The energy dissipated through hysteresis can be varied in many ways. We describe, in the sequel, many experiments in which the hysteresis loss for constant values of H_0 was caused to vary by varying the stress applied to the specimen; Mr. WILLS has varied the hysteresis loss for constant values of H_0 by varying the temperature of the specimen. But perhaps the most natural, and certainly the most usual, way of varying W is to vary H_0, while the stress is kept at a constant zero value. The curve representing W as a function of B_0 under these conditions is the curve which is useful to engineers when designing transformers. To distinguish it from the curves, which represent W as a function of B_0, when H_0 is kept constant and B_0 is varied by varying the stress, we call it a $W - B_0$ curve for zero stress. It is the curve for which Mr. C. P. STEINMETZ has proposed the formula $W = \eta B_0^{1 \cdot 6}$.

Effects of Stress.

§ 51. We now pass on to describe a number of experiments in which we studied the effects of tension and torsion upon the mean maximum magnetic induction, and upon the energy dissipated by hysteresis when the magnetic force ranged between definite limits $\pm H_0$. In each case the same series of stresses was applied to the specimen for each value taken for H_0.

Effect of Tension on Soft Iron Wire.

§ 52. The first systematic experiments on the effect of stress upon the energy dissipated in hysteresis were made in 1898 upon a soft iron wire.

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The wire was placed horizontally, perpendicular to the magnetic meridian, and the tension was applied by a flexible silk cord which passed over a pulley and supported a weight. The reversing key described in § 30 was used, the current being measured by a shunted d'Arsonval galvanometer. We satisfied ourselves by the comparison of the values of W found (1) from cyclic B—H curves, (2) by the electro-dynamometer, that the method was yielding at least approximately exact values of W. Before the observations corresponding to any given value of H_0 were made, the wire was subjected to several cycles of loading and unloading, the maximum load being 24 kilogrammes. The magnetic observations were taken only as the load was being increased. The section of the wire was ·00708 sq. centim., so that a load of 1 kilogramme gives a tension of $1 \cdot 39 \times 10^9$ dynes per sq. centim. The results are given in the following table and in fig. 7. (See also § 67.)

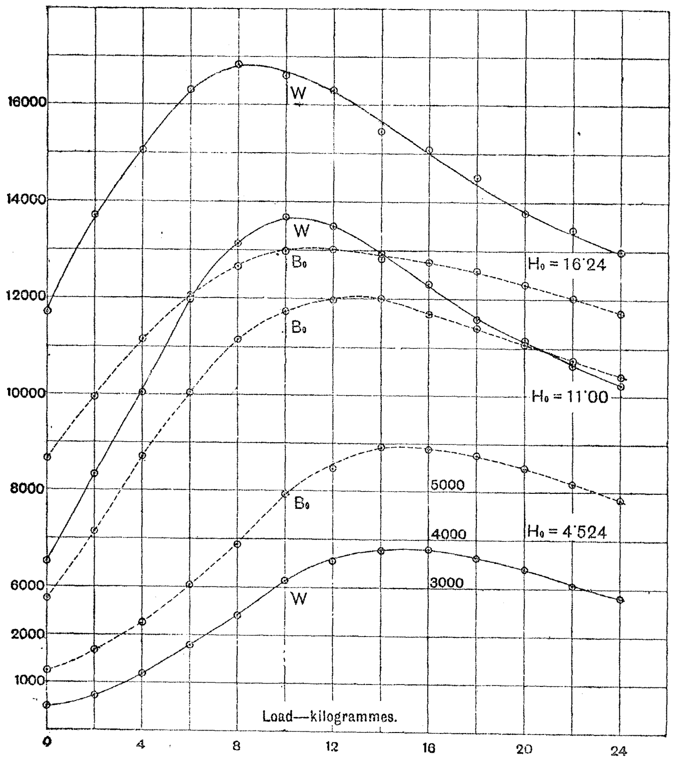

Fig. 7.

Observations on a soft iron wire during both loading and unloading showed that the curves for loading and unloading are not quite identical, though the difference between them is not large.

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Load, kilogrammes.	$H_0 = 4{\cdot}524$.		$H_0 = 11{\cdot}0$.		$H_0 = 16{\cdot}24$.	
	B_0.	W.	B_0.	W.	B_0.	W.
0	1233	494	5750	6520	8660	11710
2	1663	726	7150	8320	9940	13700
4	2250	1183	8700	10030	11150	15050
6	3030	1776	10030	11970	12050	16300
8	3880	2402	11150	13120	12650	16830
10	4915	3140	11730	13680	12980	16600
12	5475	3540	11970	13500	13010	16300
14	5920	3770	12000	12920	12820	15550
16	5870	3820	11690	12300	12760	15080
18	5750	3610	11400	11590	12580	14500
20	5490	3400	11070	11150	12300	13780
22	5160	3060	10740	10630	12010	13430
24	4840	2800	10410	10220	11720	12980

In order to save space on the diagram, the zero for the four upper curves differs from that for the two lower ones. The numbers on the diagram will prevent any confusion.

In each case, as the tension increases, both B_0 and W rise to maximum values, the tension corresponding to the maximum values diminishing as H_0 increases. Next to the similarity between the curves for B_0 and W for a given value of H_0 the most striking feature is the great increase in both B_0 and W occasioned by tension when H_0 is small. Thus for $H_0 = 4{\cdot}524$ a pull of 14 kilogrammes raises B_0 from 1233 to 5920, and W from 494 to 3770. The magnetic force was not carried to values high enough to obtain the Villari reversal of the effect of tension.

Effect of Torsion within the Elastic Limit.

§ 53. In July and August, 1899, we made a long series of experiments on the effect of torsion upon the magnetic qualities of iron and steel. The arrangements for applying torsion were very simple. A wooden wheel 24·7 centims. in diameter was mounted on a brass tube as an axle, and this tube revolved in a bearing. The wire under test passed through the solenoid and through this tube, and was clamped at one end to the wooden wheel, while the other end was held in a vice. The vice and the bearing of the wheel were mounted on a stiff wooden beam so that the wire was parallel to the beam, which was placed at right angles to the magnetic meridian. The magnetising solenoid was so fixed that the wire passed along its axis.

To apply a torsional couple to the wire, a weight was hung from the edge of the wheel by a flexible string wrapped round the wheel. When we wished to give the wire a definite twist, the wheel was clamped in the desired position.

Experiments on Steel.

§ 54. *$W - B_0$ Curve for Zero Stress.*—The first set of experiments was made upon a steel rod ·09125 sq. centim. in section. To find how B_0 and W depend upon H_0, when there is no torsion, we made a series of observations for B_0 and W, using the key of § 31, and varying H_0 from 37·4 to 5·0; the results are given in the table below. The electro-dynamometer was not sensitive enough to allow us to work with magnetic forces less than 5 units. These observations will serve as an example of the application of the method described in this paper to determine the manner in which W depends upon B_0, when B_0 is made to vary by changing H_0, the stress being constant. The curve representing these observations is shown in fig. 8. The numbers placed along the curve show the values of H_0 for which the values of B_0 and W_0 were found. The curves (1) to (4) are considered in § 66.

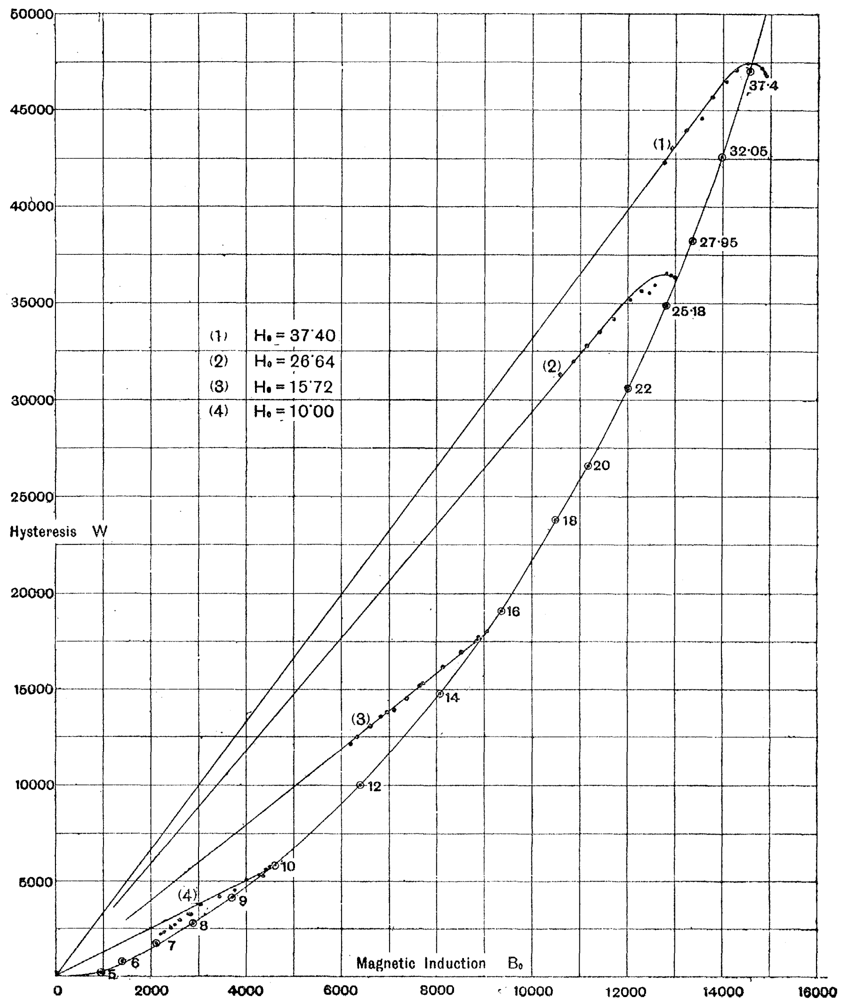

Fig. 8.

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H_0.	B_0.	W.	H_0.	B_0.	W.	H_0.	B_0.	W.
37·40	14550	47050	18·00	10480	23800	8·00	2870	2780
32·05	13980	42600	16·00	9360	19100	7·00	2110	1700
27·95	13350	38200	14·00	8080	14820	6·00	1400	780
25·18	12800	34900	12·00	6390	10020	5·00	950	190
22·00	12010	30600	10·00	4600	5830			
20·00	11160	26600	9·00	3690	4100			

§ 55. *Effect of Torsion, within the Elastic Limit, upon the Steel Rod.*—Thes teel rod was now subjected to torsion in the way described in § 53. A load not exceeding 2000 grammes was hung from the rim of the wheel and was increased and diminished by steps of 100 or 200 grammes, care being taken to make the changes in the load with as little jerking as possible. For each cycle of loading and unloading the maximum magnetic force H_0 was kept at a constant value. In every case, except possibly that of $H_0 = 37·4$, the wire was put through several cycles of twist in which the load varied between the limits $\pm$ 2000 grammes, and then the magnetising current was put through 20 cycles of reversal before the observations for B_0 and W were made. The positive and negative signs indicate that the weights were hung on the right- and left-hand sides of the wheel respectively. In the actual experiments on the effect of torsion on B_0 and W the torsion was always positive, the load being initially zero, then increasing to 2000 grammes, and then decreasing to zero. In the ideal case of perfect symmetry about the state of no torsion, the part of the curve for negative torsions would be exactly similar to the part of the curve for positive torsions. The results are given in the following table, and are exhibited in fig. 9 :—

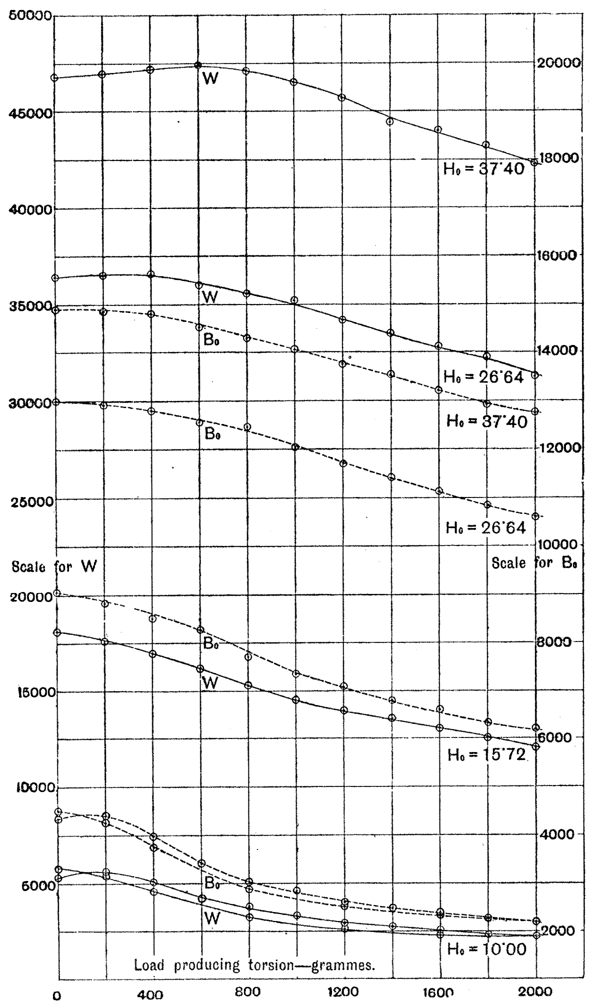

Fig. 9.

Load, grms.	$H_0 = 37{\cdot}4$.		$H_0 = 26{\cdot}64$.		$H_0 = 15{\cdot}72$.		$H_0 = 10{\cdot}0$.		$H_0 = 5{\cdot}0$.	
	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.
0	14890	46800	13000	36400	9060	18100	4350	5310	674	190
200	14860	46900	12920	36500	8840	17620	4410	5650	690	196
400	14810	47200	12800	36600	8520	16950	3980	5070	685	194
600	14520	47400	12560	36000	8130	16180	3420	4190	677	206
800	14290	47100	12460	35600	7700	15300	3020	3760	621	165
1000	14060	46500	12040	35200	7360	14560	2840	3320	628	173
1200	13760	45700	11700	34200	7100	13950	2600	2890	602	166
1400	13540	44600	11420	33500	6800	13550	2480	2730	584	151
1600	13200	44000	11130	32800	6600	12980	2390	2540	547	130
1800	12910	43200	10850	32000	6330	12510	2260	2350	547	129
2000	12760	42300	10600	31300	6210	12050	2190	2230	556	140
1600	13210	43900	11100	32700	6520	12660	2330	2310	552	126
1200	13780	45700	11700	34100	6960	13800	2510	2610	585*	144*
800	14290	47000	12300	35700	7630	15230	2870	3230	635†	173†
400	14690	47100	12800	36500	8540	17100	3750	4610	657	171
200	14850	46600	13000	36600	8860	17700	4270	5380	704	206
0	14920	46800	13050	36800	9030	18200	4510	5770	794	265

* 1000 grammes. † 600 grammes.

From $H_0 = 37{\cdot}4$ to $H_0 = 15{\cdot}72$ the curves for unloading agree, within the errors of experiment, with the curves for loading, and only a single line is shown in the diagram. In the cases of $H_0 = 10{\cdot}0$ and $H_0 = 5{\cdot}0$ the curves for unloading differ from those for loading. The curve for $H_0 = 5$ could not be conveniently shown on the diagram. In these two cases there is also an evident want of symmetry, for, if the symmetry were perfect, the lines for loading and unloading would cross each other on the line of no load. It is interesting to notice how closely the curves for W imitate those for B_0. We consider this similarity in § 66 below.

Experiments on Soft Iron.

§ 56. A much more extended series of experiments was made upon soft iron wire, ·0324 sq. centim in cross-section. The wire was "galvanised" when bought, but by heating it to bright redness in a large blowpipe flame all the zinc was burnt off the wire and at the same time the wire was annealed. The wire was twisted by means of the arrangement described in § 53. The reversing key described in § 31 was used in these experiments.

The first step was to test by the method of § 41 whether X and Y could be neglected. The numbers for this specimen, recorded in section (*b*) of § 41, show that X and Y were negligible.

§ 57. *$W - B_0$ Curve for Zero Stress.*—To gain a general idea of the magnetic character of the wire, a set of observations was made to determine B_0 and W for a series of values of H_0, the wire having been annealed and being free from strain.

The magnetic force was put through about 20 cycles of reversal before the tests for B_0 and W were made. The results are given in the following table, and the curve representing them is shown in fig. 10; the curves (1) to (5) are considered in § 65.

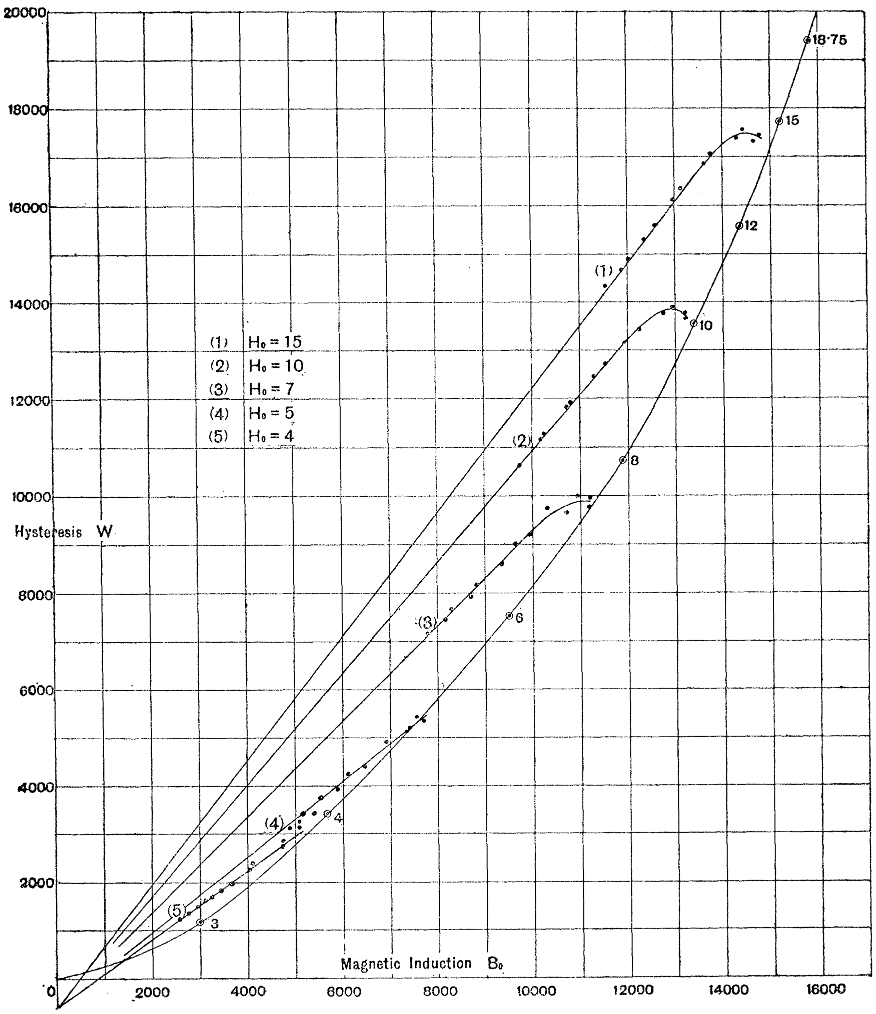

Fig. 10.

H_0.	B_0.	μ.	W.	H_0.	B_0.	μ.	W.
18·75	15800	843	19400	8·0	11880	1485	10720
15·0	15190	1012	17720	6·0	9490	1582	7530
12·0	14350	1196	15580	4·0	5660	1415	3424
10·0	13380	1338	13550	3·0	2990	997	1170

Effect of Torsion within the Elastic Limit.

§ 58. The wire was found capable of sustaining 300 grammes hung from the edge of the wheel without acquiring permanent set. We therefore decided to subject it to cycles of torsion in which the load varied within the limits ± 300 grammes. The changes in the load were made by steps of 50 grammes.

Before the cycle of loading and unloading, during which the magnetic observations were made, the wire was subjected to several cycles of positive and negative torsion, the load having the limits ± 300 grammes. In every case, before the observations for B_0 and W were made for any particular value of the torsion, the magnetic force was put through about 20 cycles of reversal; in this way we avoided the major part of the indeterminateness due to the diminution of B_0 and W which occurs initially with continued reversals of H_0. (§ 46.)

The results of these experiments are given in the table below and are exhibited in fig. 11 :—

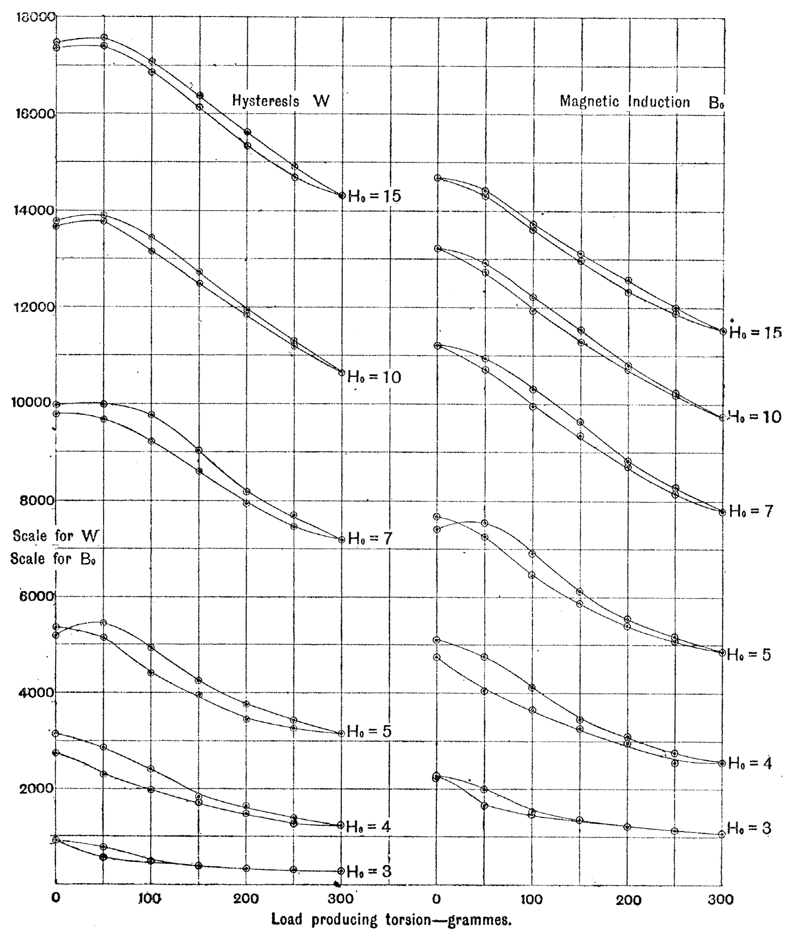

Fig. 11.

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Load, grms.	$H_0 = 15 \cdot 0$.		$H_0 = 10 \cdot 0$.		$H_0 = 7 \cdot 0$.		$H_0 = 5 \cdot 0$.		$H_0 = 4 \cdot 0$.		$H_0 = 3 \cdot 0$.	
	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.
0	14680	17470	13210	13780	11200	9970	7400	5180	5100	3140	2280	907
50	14420	17560	12920	13900	10940	9990	7540	5440	4750	2860	1980	766
100	13730	17080	12220	13440	10310	9760	6910	4920	4110	2400	1520	484
150	13120	16370	11540	12720	9640	9020	6120	4250	3440	1840	1350	389
200	12570	15610	10800	11930	8820	8180	5550	3760	3100	1630	1220	336
250	12000	14900	10240	11280	8300	7700	5170	3420	2760	1380	1140	302
300	11530	14320	9740	10630	7790	7180	4860	3135	2540	1242	1070	276
250	11880	14680	10190	11190	8150	7460	5070	3260	2550	1265	1120	286
200	12320	15330	10710	11830	8700	7940	5390	3443	2950	1480	1220	329
150	12970	16130	11280	12480	9350	8600	5870	3950	3260	1700	1320	367
100	13610	16850	11930	13150	9960	9210	6460	4405	3660	1980	1460	443
50	14300	17400	12730	13770	10700	9660	7250	5130	4040	2290	1650	563
0	14670	17350	13200	13660	11170	9770	7680	5350	4740	2745	2230	911

In each cycle of torsion both the mean maximum induction B_0 and the hysteresis W are greater when the torsion is increasing than when it is diminishing, except for small parts of those curves in which want of symmetry about the line of zero torsion has caused the crossing-point of the two branches of the curve to lie off that line.

For the smaller fields, both B_0 and W are very sensitive to torsion. Thus, when $H_0 = 3$, a load of 100 grammes hung from the edge of the wheel diminished B_0 by about one-third and W by about one-half of their values for zero load.

The dynamometer was not sensitive enough to allow us to continue the observations for W for fields less than $H_0 = 3$. We could have measured B_0 for much smaller fields, but the observations would not have been of much interest in the absence of the observations for W.

In the case of the smaller magnetic fields we noticed that the hysteresis continued to diminish considerably with continued reversals of the magnetic force even after it had been subjected to many reversals.

The table of § 57, and the top and bottom rows of the table just above, give the values of B_0 and W for zero torsion for various values of H_0. The values of B_0 and W in the first table do not agree very closely with those in the second table, but it must be noticed that the first table was made after the wire was annealed and before it was strained in any way. After the observations of § 57 had been made, experiments were made to find the limits of elasticity of the wire, and in this process the wire was subjected to torsions large enough to give it permanent set; to get rid as far as possible of the effects of this overstraining, the wire was re-annealed before the tests for the second table were made. Close agreement between the two tables is in consequence not to be expected.

These experiments may serve to show the saving of time effected by our method of measuring hysteresis, for the observations for the first four values of the magnetic force, involving fifty-two determinations of hysteresis, were easily taken in one day, though much time was spent in changing the load, in putting the current through 20 reversals before the magnetic tests were made, and in bringing the suspended coil of the dynamometer to rest.

Effect of Torsion beyond the Elastic Limit.

§ 59. So far our experiments were made for torsional strains in which the elastic limit was not overpassed at all, or in any case was not exceeded to such an extent that the wire took a noticeable permanent set. We thought it would be interesting to trace the effect of great torsional overstrain upon the induction and the hysteresis for a constant maximum magnetic force $H_0 = 5$. In 1899 we made experiments in this direction upon two specimens of the same soft iron wire as was used in the experiments of §§ 57, 58. Both specimens were heated in a large blowpipe flame to burn off the zinc coating and to anneal the wire. The arrangement of § 53 was used for applying torsion.

§ 60. *Experiments on Soft Iron Wire* (1); $H_0 = 5$.—In these experiments the wheel was turned, always in one direction, through the desired angle, measured from its position when the wire was unstrained, and was then clamped. A mechanical "counter" served to record the number of revolutions made by the wheel. When the wheel had been clamped in any position, the magnetic force was put through 20 cycles of reversal, and then readings for B_0 and W were made. The wheel was now turned still further and again clamped, and then the magnetic observations were repeated. This process was continued till the wire broke. The fracture occurred when the wheel had been turned through 104 revolutions. Since the length of the wire was 65·5 centims., this angle corresponds to a twist of 1·57 turns per centim. The observations were too numerous to be recorded in a table; we therefore only give a diagram in fig. 12, distinguishing the curves for this specimen by the mark (1).

In these experiments the first result of the torsion was to cause a very rapid diminution of both B_0 and W. The curve for B_0 falls continuously till the wire breaks, but the curve for W shows a well-marked rise and fall, with a maximum at about 7·5 revolutions. The values of B_0 and W were determined immediately after the wire broke, the tests showing that, when the stress was relieved by the fracture, both B_0 and W fell to about half the values they had just before the fracture. Thus B_0 fell from 2950 to 1510, while W fell from 2410 to 1044.

We found that when the wire had been twisted W diminished considerably with continued reversals, even after 20 cycles of magnetisation.

§ 61. *Experiments on Soft Iron Wire* (2); $H_0 = 5·0$.—The second specimen (2) was treated rather differently. The wheel was turned through a definite number of revolutions, measured from its position when the wire was initially free from torsion the number being read on the "counter." The wheel was then clamped in this position, and, after 20 cycles of magnetisation, observations were made for B_0 and W. The wheel was now unclamped, and the wire was allowed to untwist so as to rid

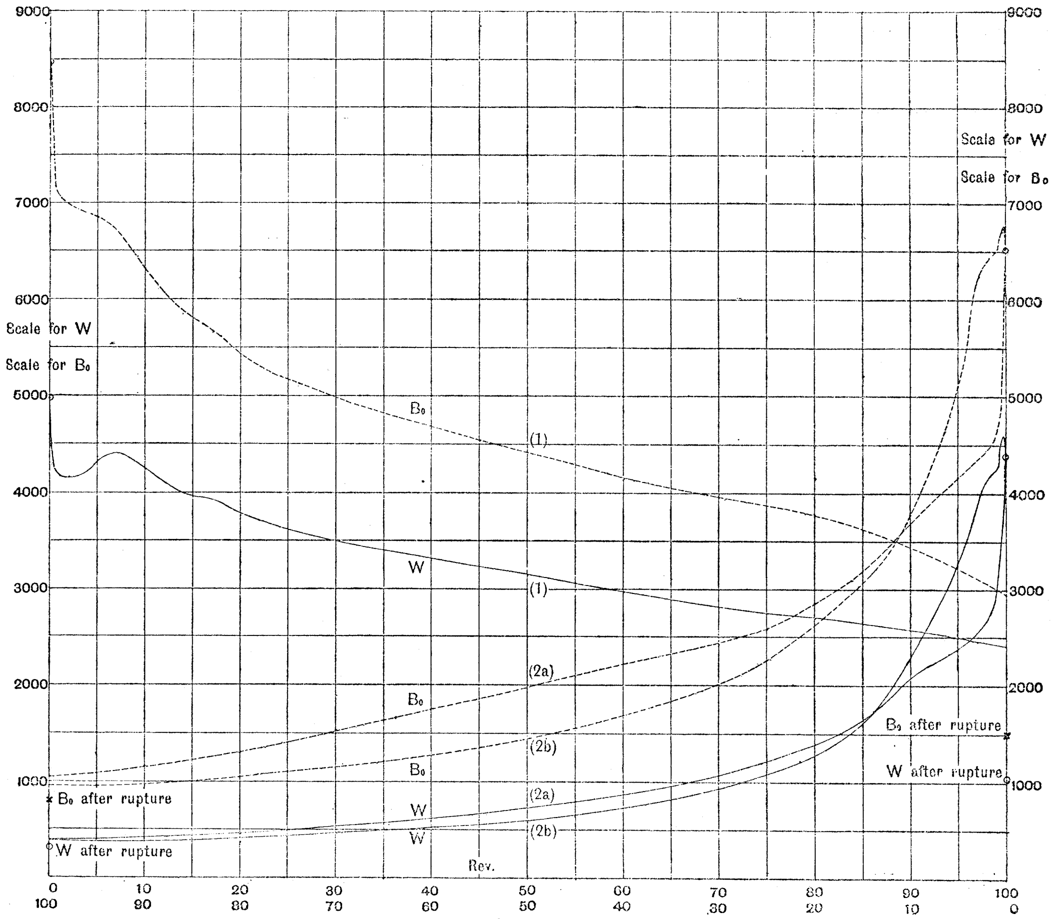

Fig. 12.

itself of torsional stress; after twenty more cycles of magnetisation, the new values of B_0 and W were then determined for this condition of the wire. The wheel was then turned still further, and fresh observations were made, the process being repeated until the wire broke. The results are shown in fig. 12. The curves for B_0 and W when the wire was under torsional stress are marked (2*a*), and those for zero stress are marked (2*b*). For the sake of clearness, the origin for these four curves is at the *right* side of the diagram.

The wire in untwisting turned the wheel back through an angle depending upon the twist from which it was endeavouring to rid itself—an angle which increased with that twist. After a twist of one revolution the wire turned the wheel back through 100°, retaining a "permanent set" of 260°, while, after a twist of 100 revolutions, the wheel was turned back through 290°. In the curves (2*b*) the points for B_0 and W, which are placed on the ordinate corresponding to any number, x, of complete revolutions, indicate the values found for B_0 and W when the wire was allowed to free itself from stress after the wheel had been turned through x revolutions.

The curves ($2a$) show that, as in the case of specimen (1), the application of torsion causes a continued decrease of both B_0 and W. The decrease is at first very rapid, B_0 falling from 6520 to 4750 and W from 4380 to 3180 for the first half revolution of the wheel. The curves ($2b$) show that after the first half revolution of the wheel both B_0 and W diminish continually, but at first much less rapidly than is the case for the curves ($2a$). The initial parts of the curves ($2b$) are an exception to the rest, for when the wire was allowed to untwist after the wheel had been turned through the first half revolution both B_0 and W were greater than before the wire was twisted at all, B_0 rising from 6520 to 6720 and W from 4380 to 4590.

Up to about 13 revolutions of the wheel, the relief of torsional stress was followed by an increase of both B_0 and W, but beyond this point the relief caused a decrease of both B_0 and W.

The wire broke at 101 revolutions of the wheel, *i.e.*, at practically the same twist as wire (1); the length was very nearly the same, viz., 65·7 centims. Just before fracture the values of B_0 and W under the torsion due to 100 revolutions were $B_0 = 1050$, $W = 383$, and when the wire had freed itself from stress, $B_0 = 960$, $W = 383$. After fracture the values fell to $B_0 = 787$, $W = 302$, the changes being much smaller than those for specimen (1). It seems probable that the diminution of B_0 and W which occurs on fracture is due to the violent jar which attends the fracture.

The values of both B_0 and W in ($2a$) and ($2b$) are for large twists very much smaller than the corresponding values in (1). It is true that the permeability of (2) before it was strained, viz., $\mu = 1304$ for $H_0 = 5$, is much less than the value $\mu = 1694$ for (1) under the same conditions. But the great difference between (1) and (2) in respect to B_0 and W for large twists can hardly be due to this fact. It is more likely due to the untwisting of the wire at the various stages of the process.

When the wire had been twisted we found just as for (1) that W diminishes considerably even after 20 cycles of magnetisation. We took 20 cycles in each case before making the tests for B_0 and W, but if W is still diminishing with continued reversals, it is hardly possible to assign any very definite value to it.

The wire, initially quite pliable, was very stiff after it had been broken by twisting. Its length originally was 65·7 centims., and this was increased by the torsion to 66·9 centims. The mean diameter diminished from ·0796 to ·0793 inch. Thus the volume of the wire increased by about 1 per cent.

The smallest values reached by B_0 and W, viz., those found after fracture, are very small compared with those found before the wire was strained. Initially $B_0 = 6520$, $W = 4380$, and finally $B_0 = 787$, $W = 302$.

Influence of Permanent Set upon the Effects of Cycles of Torsion.

§ 62. The experiments of § 61 made on specimen (2) showed that up to twists giving a permanent set of about 13 revolutions in a length of 65·7 centims. both B_0 and W become greater when the torsional stress is relieved, and that beyond the limit of 13 revolutions the reverse is the case. This observation led us to inquire into the forms taken by the two curves connecting B_0 and W with a cyclical torsional couple applied to a piece of soft iron wire, which had been previously twisted so as to have a considerable permanent set. The actual specimen was that used in § 58 without any further annealing. In carrying out these experiments, the wheel was turned through a definite number of revolutions, measured from its position before the wire was strained, thereby giving the wire a permanent set. The wheel was then set free so as to allow the wire to untwist itself as far as possible, and then about 10 cycles of twisting were given to the wire, the load producing the torsional couple varying between the limits of $\pm$ 300 grammes. After these cycles of twisting, a set of magnetic observations was taken as the next cycle of twisting was gone through. In every case the magnetic force was put through 20 cycles of reversal before the magnetic observations were made.

One effect of the cycles of twisting was to diminish the "permanent set" of the wire. Thus, reckoning from the position of the wheel when the wire was set free after a twist of 50 revolutions, 10 cycles with the load between the limits of $\pm$ 300 grammes caused the wheel to turn back through $\frac{3}{4}$ revolution.

It will be noticed that the curves in figs. 13, 14 do not always form closed figures. This is probably in the main due to the fact that the preliminary cycles of twisting were gone through in a few minutes, while the one during which the tests for B_0 and W were made occupied about an hour. We have often observed a similar effect when a copper wire is put through cycles of loading and unloading. There is in this case so much "creeping" when a load is applied, that, if a cycle of loading and unloading with a given range of load is performed slowly after a number of cycles of loading and unloading performed comparatively quickly with the same range of load, the curve connecting the elongation and the load is not closed.

The curves exhibit in a highly developed form the want of symmetry which was rudimentary in the experiments of § 58 (fig. 11).

In every case the maximum magnetic force was kept at the constant value $H_0 = 5·0$, so that the experiments might be comparable with those of § 61. The length of the wire, 65·5 centims., was about the same as in the experiments of § 61. The results are given in the following table and are exhibited in figs. 13, 14. The + sign prefixed to a load in the table indicates that the couple due to it has the same sign as the couple which gave the wire its permanent set. In order to show all the curves on a single diagram we have taken a different position of the zero line for B_0 and W in each curve. The numbers marked near each curve will enable the values of B_0 or W to be read off for any point on the curves.

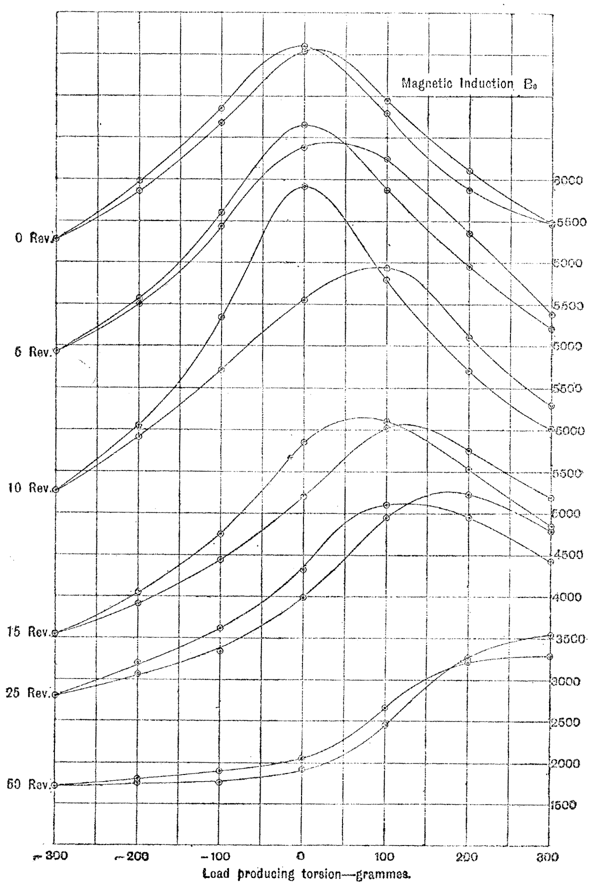

Fig. 13.

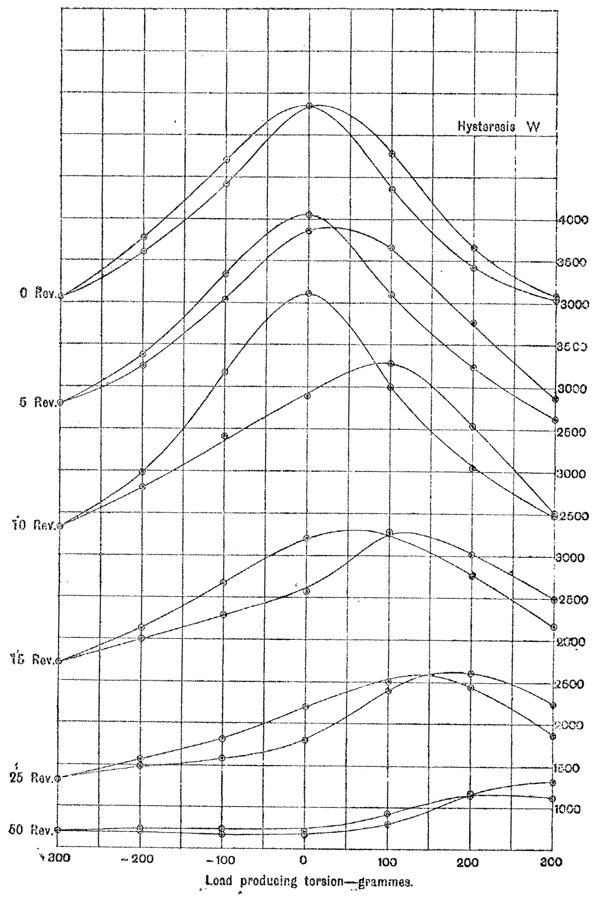

Fig. 14.

Load, grammes.	0 Rev.		5 Rev.		10 Rev.		15 Rev.		25 Rev.		50 Rev.	
	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.	B_0.	W.
+300	5460	3040	5210	2630	5020	2480	4850	2170	4430	1870	3290	1130
+200	5870	3420	5950	3240	5700	3040	5530	2760	4950	2440	3220	1150
+100	6790	4360	6870	4100	6790	4000	6100	3260	5100	2510	2660	930
0	7600	5350	7650	5050	7920	5110	5850	3200	4320	2200	2040	705
−100	6850	4700	6600	4340	6340	4170	4750	2670	3610	1810	1900	740
−200	5980	3770	5570	3370	5050	2980	4050	2130	3200	1560	1800	732
−300	5280	3060	4930	2800	4270	2330	3540	1720	2800	1320	1710	692
−200	5860	3600	5500	3240	4910	2800	3910	2000	3060	1490	1750	700
−100	6680	4420	6430	4040	5710	3410	4440	2290	3340	1580	1760	672
0	7540	5360	7380	4860	6550	3890	5200	2570	3990	1800	1920	673
+100	6940	4780	7240	4660	6930	4280	6020	3280	4950	2400	2450	802
+200	6100	3660	6350	3770	6100	3540	5750	3020	5230	2610	3260	1172
+300	5480	3090	5370	2880	5300	2510	5190	2490	4790	2240	3540	1317

The curves tend to confirm the result of § 61, viz., that for small amounts of permanent set both B_0 and W increase when the couple is removed. The sixth curve shows that when the permanent set is large the removal of the couple causes a decrease of both B_0 and W. In these experiments the change from increase to decrease occurs somewhere between 25 and 50 revolutions, as against about 13 in the experiments of § 61. The wire of § 61 was not, however, subjected to cycles of twisting.

The curves for zero permanent set correspond to the curves for $H_0 = 5$ found in § 58 and shown in fig. 11. But instead of curves for only half a cycle of twisting we have now curves for a complete cycle. The mean values of B_0 and W for zero couples are very nearly the same in the two cases, but the couple due to a load of 300 grammes produces a decrease of B_0 and W rather greater than was found in the experiment of § 58 for $H_0 = 5$. The specimen had, however, been put through many cycles of twisting in the interval.

The observations could not be extended beyond 50 turns of permanent set, for when the attempt was made the wire broke at 54 turns. The similar wires used in the experiments of §§ 60, 61 did not break till 104 and 101 revolutions respectively. It is possible that the difference, if not due to a flaw, is due to the effect of the cycles of positive and negative couples.

The curves of figs. 13, 14 show in a very striking manner the close correspondence between W and B_0 when H_0 is kept constant and B_0 is made to change by varying the stress.

Development of a Cyclic State after Initial Permanent Set.

§ 63. In the last series of experiments made in 1899 we studied the manner in which a wire, after torsional overstrain, gradually settles down to a definite cyclic state as the applied couple goes through a series of cycles. The specimen was a piece of the same annealed soft iron wire as that used in §§ 58, 60, 61, and had the same length as the wire used in § 60, viz., 65·5 centims. In these experiments the wire was strained far beyond the elastic limit by turning the torsion wheel through a definite number of revolutions—in our experiments 6, 20, and 50—from the position for zero strain. The wheel being now clamped, a weight of 300 grammes was hung from the right edge of the wheel, producing a couple having the same sense as that which strained the wire. The wheel was then unclamped, and the wire, being allowed to untwist, turned the wheel back until the couple due to the elasticity of the wire balanced the couple due to the 300 grammes. After 20 cycles of magnetisation, H varying between the limits $\pm$ 5, B_0 and W were determined. The load was then reduced to 200 grammes, and the magnetic tests were repeated. When by two more steps of 100 grammes the load had been reduced to zero, loads of 100, 200, and 300 grammes were hung in succession from the left edge of the wheel to produce negative couples. The process of changing the load from + 300 to − 300 grammes and back again was continued till it appeared that the curves connecting B_0 and W with the load had nearly settled down to a permanent form.

In the first experiments, when the permanent set was about 6 revolutions, the first application of a *negative* couple due to 200 grammes caused the wire to yield somewhat, and when the load of −300 grammes was reached, the yielding was great enough to cause the wheel to turn gradually through half a revolution. In the next cycle of loading this effect of a negative couple was small, and it became much smaller for each successive cycle. With the large permanent sets of 20 and 50 revolutions the wire became so hard that the yielding under a negative couple was very small even in the first cycle. The positive couples caused little yielding even in the first cycle with the permanent set of 6 revolutions.

The observations were too numerous to be recorded, except in the form of the curves in figs. 15, 16. In order to show all the curves on a single diagram, we have taken a different position of the zero line for B_0 and W in each curve.

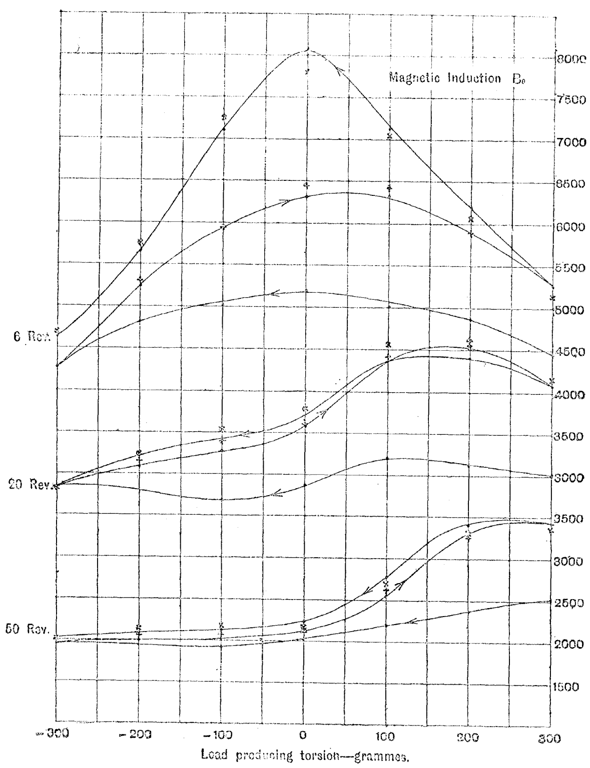

Fig. 15.

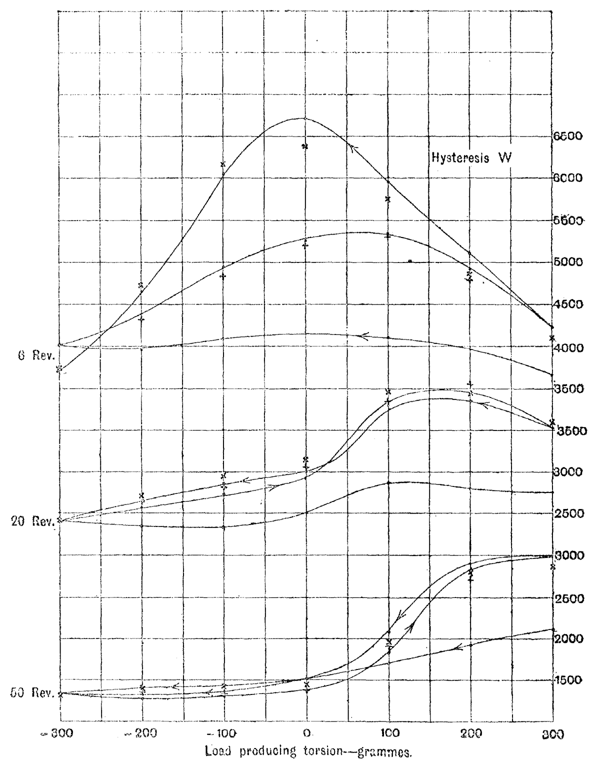

Fig. 16.

In these experiments it was found that after the first application of a negative couple due to 300 grammes the iron had roughly reached a cyclic state, and thus, to avoid confusion, we give only $1\frac{1}{2}$ cycles as continuous curves. For a permanent set of 6 turns $3\frac{1}{2}$ cycles were actually gone through, and in each of the other cases

$2\frac{1}{2}$ cycles. In each case the last cycle is indicated by isolated points; when the positive load is increasing a + is used, and when the positive load is decreasing a × is used.

In their general forms the curves due to the last cycle for 20 and 50 turns of permanent set resemble those obtained after several comparatively rapid cycles of torsion for 25 and 50 turns of permanent set in § 62. Any close comparison is impossible on account of the fact that the curves of § 62 do not form closed figures.

In the case of 6 turns of permanent set, the curves for the last cycle differ widely from those for 5 turns of permanent set in the experiments of § 62. The part of the curve which corresponds to a decreasing positive couple now lies always above that part which corresponds to an increasing positive couple instead of crossing it. The two parts of the curve show, however, a tendency to approach each other with repetition of the cycles of torsion.

In this case there is a very great recovery of magnetic quality in the first $1\frac{1}{2}$ cycles of torsion. When the load was reduced to zero for the first time B_0 was 5190, at the second zero load we found $B_0 = 6310$, and at the third zero load $B_0 = 8050$. Since $H_0 = 5$, the permeability rose from 1038 to 1610.

Relation connecting W with B_0 and H_0.

§ 64. A cursory examination of the curves recording the effect of strain upon the hysteresis and the induction, for given values of the magnetic force, is sufficient to make it evident that the changes in B_0, due to strain, are very closely followed by the consequent changes in W. This correspondence is so close that it invites the attempt to express it mathematically. We have here the means of analysing W and determining it as a function of H_0 and B_0; this is impossible with the $W - B_0$ curve for zero stress, since we cannot, without straining the specimen, change B_0 without changing H_0. Our plan has been to plot curves showing W in terms of B_0 for definite values of H_0, the variations of B_0 and W being due to strain.

§ 65. The first example is taken from the experiments on the effect of torsion upon a soft iron wire, the corresponding values of B_0 and W_0 for definite values of H_0 being recorded in § 58. Five curves connecting W and B_0 were plotted for five values of H_0 upon fig. 10, where the $W - B_0$ curve for zero stress is also shown. The points corresponding to increasing torsion lie so closely on the same curves as those corresponding to diminishing torsion, that we have marked both sets in the same manner. For the larger values of H_0 each curve consists of a straight line with a small hook at one end—the end corresponding to zero strain. The straight parts of the curves all pass, on prolongation, through the point -600 on the axis of W, and hence over the greater part of the variation of B_0, caused by torsion under a constant maximum magnetic force H_0, the value of W is given by

$$W = mB_0 - 600,$$

where m is a function of H_0. To determine m in terms of H_0 we plotted m against H_0, and found that m is nearly proportional to $H^{\frac{1}{2}}_0$ as the following table shows :—

H_0	4	5	7	10	15
m	·71	·785	·99	1·15	1·28
$·35H^{\frac{1}{2}}_0$ . . .	·70	·78	·925	1·10	1·35

Hence, with fair accuracy, the formula

$$W = ·35H^{\frac{1}{2}}_0 B_0 - 600$$

represents W as a function of H_0 and B_0, over a large range, provided that the load producing torsion exceeds 100 grammes. The hooked part of each curve corresponds to small torsions due to loads of 100 grammes and under. The value of W for zero strain, recorded in § 57, is given with considerable accuracy by $W = ·326\, H^{\frac{1}{2}}_0 B_0 - 320$ over a considerable region in the neighbourhood of the maximum permeability.

§ 66. The second example is furnished by the experiments on the torsion of a steel rod, described in § 55. The curves connecting W and B_0 are shown in fig. 8 along with the $W - B_0$ curve for zero stress. We again find that for the larger values of H_0 those portion of the curves which correspond to the larger stresses are straight lines radiating from a single point—in this case the origin—and thus obtain $W = mB_0$. The values of m and H_0 are given in the table.

H_0	10	15·72	26·64	37·4
m	1·25	1·98	2·94	3·32
$·64\,(H_0 - 6·2)^{\frac{1}{2}}$ . .	1·25	1·97	2·90	3·57

Thus, approximately,

$$W = ·64\,(H_0 - 6·2)^{\frac{1}{2}} B_0 .$$

This expression would naturally fail to represent facts when $H_0 < 6·2$.

The value of W for zero stress, recorded in § 54, is given closely over the whole range by $W = ·57H^{\frac{1}{2}}_0 B_0 - 1800$.

§ 67. We now take the experiments on the effect of tension upon a soft iron wire described in § 52. The curves connecting W with B_0 are shown in fig. 17 along with the $W - B_0$ curve for zero stress. Each of the curves, which shew the effects of stress, is again made up of a straight part and a hook, the straight parts radiating from the point $B_0 = -600$, $W = -950$. For the slope of the lines we have

H_0	4·524	11·00	16·24
m	·741	1·20	1·36
$·353H^{\frac{1}{2}}_0$ . . .	·750	1·17	1·42

so that the straight parts are given by

$$W = ·353H^{\frac{1}{2}}_0 (B_0 + 600) - 950 .$$

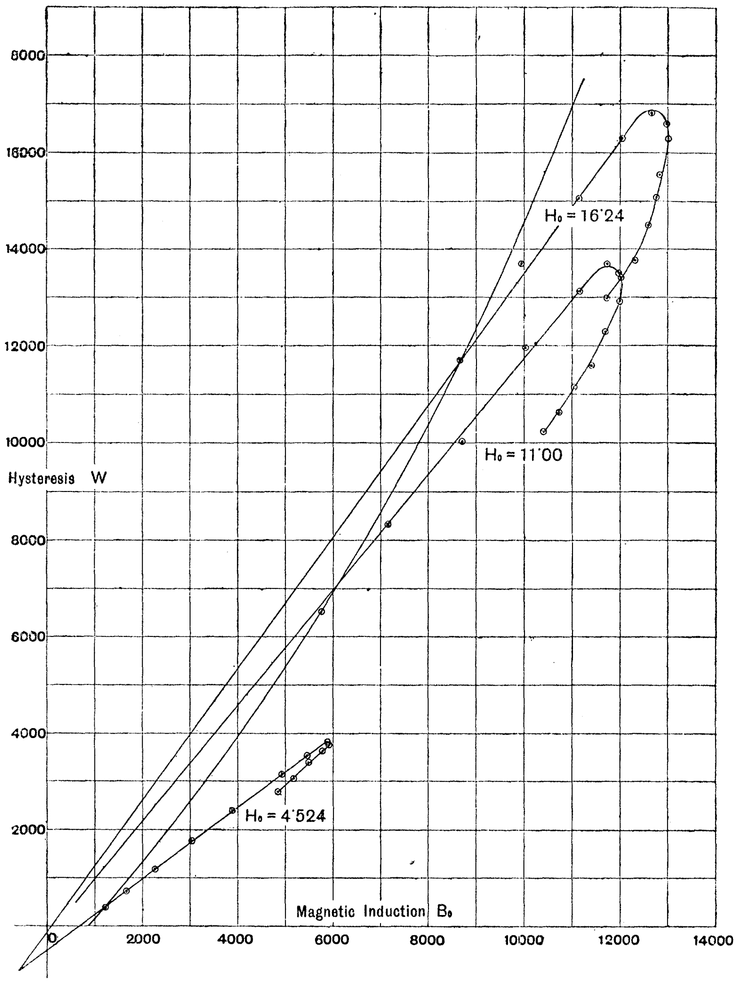

Fig. 17.

The curves just described differ in two important particulars from the curves for torsion in fig. 10. In the case of torsion the straight parts of the curves correspond to large stresses and the hooks to small stresses, but in the case of tension the straight parts correspond to small stresses and the hooks to large stresses; the hook is moreover much more developed in the curves for tension than in the curves for torsion. The second point of difference is that the tension curves lie to the right of the $W - B_0$ curve for zero stress whereas the torsion curves lie to the left.

§ 68. So far the strains have been practically within the elastic limit. We now examine some cases in which this limit was much exceeded.

In the W—B_0 curve, plotted from the experiments described in § 60, the points appear to be irregularly placed till the torsion wheel has made about eight revolutions. For greater strains the points lie well on the straight line, $W = \cdot 610 B_0 + 450$. Since $H_0 = 5$ this may be written

$$W = \cdot 273 H^{\frac{1}{2}}_0 B_0 + 450 .$$

In the experiments of § 61 the points corresponding to both the curves (2a) and (2b) cluster round a single curve, which when B_0 exceeds 2500 is represented by

$$W = \cdot 352 H^{\frac{1}{2}}_0 B_0 - 850 \qquad (H = 5).$$

§ 69. We are not prepared to offer any physical explanation of the formula $W = aH^{\frac{1}{2}}_0 B_0 - b$*. But as it has a rational appearance it seemed worth while to test it on some results obtained under zero stress. We plotted W against $H^{\frac{1}{2}}_0 B_0$ for several of the tables given by Professor J. A. EWING and Miss KLAASSEN,† as well as for the results for cobalt obtained by Professor J. A. FLEMING.‡ In every case the resulting curve was straight over a considerable range of B_0. For small values of B_0 the value of W given by $W = aH^{\frac{1}{2}}_0 B_0 - b$ is too small, while for large values of B_0 it is too large. The value of $aH^{\frac{1}{2}} B_0 - b$ begins to be too large when with increasing B_0 the permeability μ begins to fall rapidly below its maximum value. When, with decreasing B_0, μ falls much below its maximum value, $aH^{\frac{1}{2}} B_0 - b$ becomes too small. It also appears that when B_0 is less than B'_0, the value corresponding to the maximum of μ, μ may, without causing serious error, differ much more from its maximum value than when B_0 exceeds B'_0. The following examples from the experiments of Professor EWING and Miss KLAASSEN will serve as illustrations.

* [November 4, 1901. Mr. WILLS has obtained from his experiments a series of curves showing how W, for constant values of H_0, depends upon B_0, when B_0 is varied by varying the temperature. These curves exhibit in a striking manner the characteristic features of our own curves. Thus the curve for a given value of H_0 consists of a straight line and a hook, while the straight lines, corresponding to different values of H_0, all radiate from a point on the axis of W. Mr. WILLS, however, finds that the index of H_0 is $\frac{3}{4}$ instead of $\frac{1}{2}$. The relation deduced from his experiments is thus $W = aH^{\frac{3}{4}}_0 B_0 - b$.]

† "On the Magnetic Qualities of Iron," 'Phil. Trans.,' A, vol. 184, p. 985.

‡ "The Magnetic Hysteresis of Cobalt," 'Proc. Physical Soc.,' vol. 16, p. 519.

Ring II. Fine steel wire.* $W = \cdot560H^{\frac{1}{3}}_0B_0 - 1120$.					Ring IV. Thin sheet iron.† $W = \cdot314H^{\frac{1}{3}}_0B_0 - 225$.				
H_0.	B_0.	μ.	W (obs.).	W (cal.).	H_0.	B_0.	μ.	W (obs.).	W (cal.).
3·10	590	190	125	−538	1·400	420	300	59	−69
3·93	925	236	307	−90	1·687	677	402	134	51
4·08	1180	290	540	213	1·884	927	491	225	174
4·76	1820	382	1130	1102	2·32	1800	776	660	635
6·12	3960	647	4310	4370	3·24	4160	1285	2220	2125
7·48	6170	819	8300	8330	4·01	5710	1424	3440	3367
8·91	8090	907	12820	12390	4·93	7250	1470	4980	4825
10·98	10190	927	18100	17780	6·45	9030	1400	6940	6975
13·66	12070	983	23460	23880	8·89	10880	1225	9750	9965
21·89	15040	688	34330	38280	12·99	12640	975	12670	14075
32·42	16720	516	41320	52080	17·20	13760	800	14830	17695
43·91	17680	403	45770	64500	23·61	14720	622	16670	22225

Effect of an Electric Current upon Hysteresis.‡

§ 70. If an electric current be sent along an iron wire it produces a circular magnetic force which tends to link together the magnetic molecules in circular chains. We may expect that this linking of the molecules will make them less susceptible to the influence of a longitudinal magnetic force H, and that in consequence, for a given range $\pm H_0$ both the range of the longitudinal component, B, of the magnetic induction and also the energy dissipated by hysteresis in each cycle would be diminished. The effect deserves a systematic investigation, but this up to the present we have not been able to carry out. We must content ourselves with recording some qualitative experiments which show that the expected effect of an electric current actually occurs.

The experiments were made in March, 1896, by one of us with the help of Mr. John Talbot. An iron wire about 1 millim. in diameter was used, and an alternating current was employed in the primary circuit, giving rise to a *steady* deflexion of the dynamometer coil proportional to the energy dissipated by hysteresis in each cycle. In an experiment made on March 9, a steady current varying from 0 to 1·123 amperes was sent through the wire, with the result that the hysteresis was diminished. No observations were made to determine H_0, B_0, or W in absolute measure, and thus we can only represent W by means of the deflexion of the dynamometer. The table shows the result of the experiment, the third column recording the percentage diminution in the hysteresis occasioned by the passage of the current. It will be seen that the strongest current diminishes the hysteresis by nearly one-quarter.

* *Loc. cit.*, p. 995. † *Loc. cit.*, p. 1002.

‡ The experiments of Gerosa and Finzi are described by Professor Ewing, 'Magnetic Induction in Iron, &c.,' 3rd Edition, p. 330.

Current (amperes).	Deflexion.	Diminution, per cent.
0	141	0
·143	139	1·42
·212	137	2·84
·350	132	6·39
·518	127	9·93
·664	122	13·5
1·123	109	22·7

In an experiment made on March 7 the wire was subjected to tension due to a load varying from 0 to 20 kilogrammes. The deflexion due to hysteresis was observed (1) when no current flowed through the wire, and (2) when a current of definite strength flowed through the wire. The current was furnished by a single Daniell cell, and, judging from the last experiment, was about 1·3 amperes. In the table the last column shows the percentage diminution of the hysteresis due to the passage of the current. The numbers in this column are rather irregular, but they show quite clearly that the effect of the current diminishes as the tension increases. The initial increase of W and its subsequent decrease, noticed in detail in § 52, are well shown in this experiment.

Tension (kilos.).	Deflexion, current off.	Deflexion, current on.	Diminution, per cent.
0	73	55	24·7
4	90	76	15·6
8	98	82	16·3
12	94	82	12·8
16	84	74	11·9
20	75	67	10·7

§ 71. In concluding this paper we desire to express our thanks to Professor J. J. Thomson for his encouragement during the progress of the experiments, as well as for the use of the resources of the Cavendish Laboratory. We are also indebted to Mr. John Talbot, of Trinity College and to Mr. W. G. Frazer, Fellow of Queens' College, for valuable assistance during the earlier stages of the work, and to Mr. L. N. G. Filon, of King's College, for help in connexion with Appendix I. We have to thank Dr. R. S. Clay, of St. John's College, for some help in the preliminary experiments. We gladly record our obligation to the writings of Mr. Oliver Heaviside, for it was by the method of operators, so fruitfully used by him, that we first obtained the complete theory of the method. Our thanks are also due to Mr. W. G. Pye and to Mr. F. Lincoln, the mechanical assistants at the Cavendish Laboratory, for help and advice on many occasions. Their mechanical skill has been of great service to us.

Appendix I.

On the Heat produced by Eddy Currents in a Rod of Circular Section.

§ 72. The problem cannot be completely solved unless the permeability, μ, is independent of the magnetic force, a condition not fulfilled with actual specimens of iron. Though this is so, we can obtain useful information from the complete solution when μ is constant.

If the magnetic force be parallel to the axis of the rod, and if u be the current at a distance r from the axis, and if σ be the specific resistance, then the field within the rod has the characteristics

$$\sigma d(ru)/dr = -\mu r d\mathrm{H}/dt \quad . \quad . \quad . \; (1), \qquad d\mathrm{H}/dr = -4\pi u \quad . \quad . \quad . \quad . \; (2),$$

so that

$$\frac{1}{r} d(r d\mathrm{H}/dr)/dr = 4\pi\mu/\sigma \,.\, d\mathrm{H}/dt = g d\mathrm{H}/dt \quad . \quad . \quad . \quad . \quad . \quad . \; (3).$$

Expressing H in the form

$$\mathrm{H} = h_0 + h_1 r + h_2 r^2 + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \; (4),$$

where h_0, h_1, . . . are functions of t only, we see that $h_1 = 0$, since $u = 0$ when $r = 0$. Using this value of H in (3), and comparing coefficients of powers of r, we find h_2, h_3, . . . , and thus obtain

$$\mathrm{H} = \left\{1 + \frac{gr^2}{2^2}\frac{d}{dt} + \frac{g^2 r^4}{2^2 . 4^2}\frac{d^2}{dt^2} + \ldots\right\} h_0 .$$

Hence for H_a, the magnetic force at the surface where $r = a$,

$$\mathrm{H}_a = \left\{1 + \frac{ga^2}{2^2}\frac{d}{dt} + \frac{g^2 a^4}{2^2 . 4^2}\frac{d^2}{dt^2} \ldots\right\} h_0 ,$$

so that

$$\mathrm{H} = \left\{1 + \frac{gr^2}{2^2}\frac{d}{dt} + \ldots\right\}\left\{1 + \frac{ga^2}{2^2}\frac{d}{dt} + \ldots\right\}^{-1} \mathrm{H}_a$$

$$= \mathrm{H}_a - \frac{g}{2^2}(a^2 - r^2)\frac{d\mathrm{H}_a}{dt} + \frac{g^2}{64}(3a^4 - 4a^2 r^2 + r^4)\frac{d^2\mathrm{H}_a}{dt^2} + \ldots$$

Hence

$$u = -\frac{1}{4\pi}\frac{d\mathrm{H}}{dr} = -\frac{1}{4\pi}\left\{\frac{gr}{2}\frac{d\mathrm{H}_a}{dt} - \frac{g^2}{16}(2a^2 r - r^3)\frac{d^2\mathrm{H}_a}{dt^2} + \ldots\right\} \quad . \quad . \quad . \; (5).$$

Thus when H_a is known as a function of t, u can be found at any point of the section.

Now, if we had assumed that $d\mathrm{H}/dt$ is constant over the section, we should have found from (1), since $u = 0$ when $r = 0$,

$$u = -\frac{1}{4\pi}\frac{gr}{2}\frac{d\mathrm{H}}{dt} .$$

This assumption is thus equivalent to the assumption that the second, third, . . . terms in (5) are negligible in comparison with the first term. In this case we find for the rate at which heat is generated by eddy currents

$$\pi a^2 \frac{dX}{dt} = \sigma \int_0^a u^2 2\pi r dr = \int_0^a \frac{g^2 r^2}{64\pi^2} \left(\frac{dH}{dt}\right)^2 2\pi r dr = \frac{\sigma g^2 a^4}{128\pi} \left(\frac{dH}{dt}\right)^2,$$

where dX/dt has the meaning assigned to it in § 8. Here, since dH/dt is constant over the section, we may put $\mu = dB/dH$, and thus

$$\frac{dX}{dt} = \frac{A}{8\pi\sigma} \left(\frac{dB}{dt}\right)^2 = \frac{QA}{\sigma} \left(\frac{dB}{dt}\right)^2.$$

Hence, with the notation of (13) § 9, $Q = 1/8\pi = \cdot 03979$.

The second term of (5) will be negligible in comparison with the first, provided that $\pi\mu a^2/\sigma \,.\, d^2H_a/dt^2$ is negligible in comparison with dH_a/dt, and the third term will be negligible in comparison with the second if $\pi\mu a^2/\sigma \,.\, d^3H_a/dt^3$ is negligible in comparison with d^2H_a/dt^2.

Now, as in § 15, the characteristic of H_a is

$$K dH_a/dt + RH_a = 4\pi NE.$$

Hence, supposing that K may be treated as constant,

$$K d^2H_a/dt^2 + R dH_a/dt = 0.$$

Thus we see that the ratio of each term in (5) to the term before it is small, provided that $\pi\mu a^2/\sigma$ is small compared with the "time constant" K/R. When this condition is satisfied, we may treat dH/dt and also dB/dt as constant over the section of the rod, and may then calculate the eddy current from (1).

On the Heat produced by Eddy Currents in a Rod of Rectangular Section.

§ 73. The section of the rod is supposed so small that the current at any point may be calculated by FARADAY's law, on the assumption that dB/dt has the same value at all points of the section. We see by the case of the circular rod that this assumption is legitimate, provided that $\pi\mu r^2/\sigma$ is small compared with K/R, r being the radius of the largest circle inscribable in the section.

Let, now, a, b be the sides of the rectangular section, and let the origin be at the centre of the section. Then, since the magnetic force is parallel to the axis of the rod, we have, under the specified conditions

$$\frac{du}{dy} - \frac{dv}{dx} = q \quad . \quad . \quad . \quad (1), \qquad \frac{du}{dx} + \frac{dv}{dy} = 0 \quad . \quad . \quad . \quad (2),$$

$$u = 0 \text{ when } x = \pm \tfrac{1}{2}a \quad . \quad . \quad . \quad (3), \qquad v = 0 \text{ when } y = \pm \tfrac{1}{2}b \quad . \quad . \quad . \quad (4),$$

where u, v are the components of the current, and $q\sigma = dB/dt$.

Now (2) is satisfied if we write

$$u = d\phi/dy, \qquad v = -d\phi/dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

while (1) now becomes $$d^2\phi/dx^2 + d^2\phi/dy^2 = q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

The solution of (6), appropriate to the problem in hand,* is

$$\phi = -\frac{16q}{\pi^2}\Sigma\Sigma(-1)^{m+n}\frac{\cos(2m+1)\pi x/a \,.\, \cos(2n+1)\pi y/b}{(2m+1)(2n+1)\{(2m+1)^2\pi^2/a^2+(2n+1)^2\pi^2/b^2\}} \quad . \quad . \quad (7).$$

This value of ϕ satisfies $\nabla^2\phi = q$, because, m and n both ranging from 0 to ∞,

$$1 = \frac{4}{\pi}\Sigma(-1)^m\frac{\cos(2m+1)\pi x/a}{2m+1} = \frac{4}{\pi}\Sigma(-1)^n\frac{\cos(2n+1)\pi y/b}{2n+1}$$

within the limits $x = \pm\frac{1}{2}a$, $y = \pm\frac{1}{2}b$.

Now by (5) and (7)

$$u = \frac{d\phi}{dy} = \frac{16q}{\pi^2}\Sigma\Sigma(-1)^{m+n}\frac{\cos(2m+1)\pi x/a \,.\, \sin(2n+1)\pi y/b}{(2m+1)(2n+1)\{(2m+1)^2\pi^2/a^2+(2n+1)^2\pi^2/b^2\}}\,\frac{(2n+1)\pi}{b},$$

with a similar expression for v. These expressions satisfy (3) and (4), and thus all the conditions are fulfilled.

The rate at which heat is generated is given by

$$ab\,d\mathrm{X}/dt = \sigma\iint(u^2+v^2)\,dxdy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

The necessary integrations are easily effected, for the integral

$$\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\cos(2h+1)\pi x/a \,.\, \cos(2k+1)\pi x/a \,.\, dx$$

is zero unless h and k are equal. When $h = k$, its value is $\frac{1}{2}a$. Similar results hold when two sines are substituted for the two cosines.

We thus obtain

$$\frac{d\mathrm{X}}{dt} = \frac{64q^2\sigma a^2}{\pi^6}\Sigma\Sigma\frac{1}{(2m+1)^2(2n+1)^2\{(2m+1)^2+(2n+1)^2a^2/b^2\}}.$$

It is not convenient to calculate $d\mathrm{X}/dt$ from this double series, on account of the slow convergence. We therefore transform it into a single series. Now

$$\cosh\pi z/2 = (1+z^2)(1+z^2/3^2)(1+z^2/5^2)\ \ldots$$

Differentiating the logarithm of both sides, we obtain

$$\frac{\pi}{2}\tanh\frac{\pi z}{2} = 2z\Sigma\frac{1}{(2m+1)^2+z^2}. \qquad (m \text{ from } 0 \text{ to } \infty).$$

* The method of solution was suggested by notes taken by one of us at a course of lectures on hydrodynamics, given by Mr. R. A. Herman, Fellow of Trinity College, in November, 1889.

Expanding in powers of z and comparing the coefficients of z and of z^3, we find

$$\pi^2/8 = \Sigma\,(2m+1)^{-2}, \qquad \pi^4/96 = \Sigma\,(2m+1)^{-4}.$$

Hence
$$\frac{\pi^2}{8z^2} - \frac{\pi}{4z^3}\tanh\frac{\pi z}{2} = \Sigma\frac{1}{(2m+1)^2\{(2m+1)^2+z^2\}}.$$

Now put $z = (2n+1)\,a/b$, and then

$$\frac{dX}{dt} = \frac{64q^2\sigma a^2}{\pi^6}\Sigma\frac{1}{(2n+1)^2}\left\{\frac{\pi^2b^2}{8(2n+1)^2a^2} - \frac{\pi b^3\tanh(2n+1)\,\pi a/2b}{4(2n+1)^3a^3}\right\}$$
$$= \frac{ab}{\sigma}\left(\frac{dB}{dt}\right)^2\left\{\frac{b}{12a} - \frac{16b^2}{\pi^5a^2}\Sigma\frac{\tanh(2n+1)\,\pi a/2b}{(2n+1)^5}\right\}.$$

Writing this in the form $dX/dt = QA(dB/dt)^2/\sigma$, we have

$$Q = \frac{b}{12a} - \frac{16b^2}{\pi^5a^2}\Sigma\frac{\tanh(2n+1)\,\pi a/2b}{(2n+1)^5} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

As a/b increases, the expression (9) very rapidly tends to the limit

$$Q = \frac{b}{12a} - \frac{16b^2}{\pi^5a^2}\Sigma\frac{1}{(2n+1)^5} = \frac{b}{12a} - \cdot05255\frac{b^2}{a^2} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

[since $\Sigma(2n+1)^{-5} = 1\cdot0045$], the error not exceeding 1 in 4000 when a/b is as small as 2. We give a table of the values of Q for small values of a/b; it will be seen that Q is rather smaller for a square than for a circular rod.

a/b.	1.	1·5.	2.	2·5.	3.	4.	8.	16.	32.	64.
Q	·03512	·03260	·02853	·02492	·02194	·01755	·00960	·00500	·00255	·00129

Mr. L. N. G. FILON has kindly verified our results, employing the mathematical method used by DE ST. VENANT in finding the torsional rigidity of a rectangular prism. (*Cf.* THOMSON and TAIT, 'Nat. Phil.,' Part II., p. 248.)*

APPENDIX II.

On the Demagnetising Force due to Rods of Finite Length.

§ 74. In order to find how the demagnetising force at the centre of a long cylindrical rod depends upon the induction at the centre of the rod, the following experiments were made. The magnetising solenoid was placed at right angles to the

* [For an elliptic cylinder, axes $2a$, $2b$, $\phi = \frac{1}{2}q\,(x^2/a^2 + y^2/b^2)\,a^2b^2/(a^2+b^2)$ satisfies (6) as well as the condition that no current crosses the bounding surface.

We hence find $Q = ab/\{4\pi\,(a^2+b^2)\}$.—*December* 26, 1901.]

magnetic meridian, and a magnetometer was so adjusted that its magnet was vertically above the centre of the solenoid, and as near to the solenoid as possible. On sending a current through the solenoid a small deflexion of the magnet ensued, due to the finite length of the solenoid; by adjusting the wires connecting the solenoid to the rest of the apparatus this effect was easily annulled. On now placing the rod inside the solenoid there was a deflexion of the magnet, due entirely to the distribution of magnetism on the bar. When the diameter of the rod and the distance of the magnetometer needle from the rod are both small compared with the length of the rod, the magnetic force at the magnetometer, due to the rod, differs very little from the magnetic force at the centre of the rod due to the rod itself. Under these conditions the deflexion of the magnetometer may be taken as a nearly exact measure of the demagnetising force due to the rod.

When the specimen was a single thin wire, a mirror magnetometer was used; for bundles of wire, a mirror magnetometer had too small a range, and then a magnetometer, with a pointer moving over a circular scale, was employed.

If θ be the deflexion of the magnetometer, and M be the earth's horizontal magnetic force, then the demagnetising force h is given by $h = \mathrm{M} \tan \theta$.

§ 75. When iron is tested for permeability, the specimen, previously demagnetised "by reversals," is subjected to a magnetic force which is reversed many times between the limits $\pm \mathrm{H}_0$. The maximum induction B_0 is then determined from the galvanometer "throw" due to a single reversal of the magnetic force between the same limits. Starting from a small value, H_0 is increased by suitable steps, and the value of B_0 corresponding to each value of H_0 is determined, the process yielding a single point on the B_0—H_0 diagram for any given value of H_0.

We made a test of this kind upon an annealed soft iron wire, determining also, by the method of § 74, the demagnetising force h_0 for every value of H_0 employed. The area of section of the wire was ·00412 sq. centim., and its length was 48·5 centims. As the solenoid was 60 centims. in length and of small diameter, the magnetic force due to the current was very nearly uniform over the whole length of the specimen. A mirror magnetometer was used to determine h_0, the needle being about 3 centims. from the wire.

The results of this experiment are shown graphically in fig. 18. The abscissæ of the curves marked H_0, μ and h represent the values of B_0, the mean maximum induction at the centre of the wire, while the ordinates represent the corresponding values of H_0, μ $[= \mathrm{B}_0/\mathrm{H}_0]$ and h_0 respectively. The nearly straight line passing through the origin indicates the value h_0 would have if the flow of induction from the wire occurred entirely at its ends, in which case the "poles" would be concentrated at the ends of the wire. This line represents $h_0 = 2\mathrm{I}_0\mathrm{A}/l^2$, where $\mathrm{I}_0 \, [= (\mathrm{B}_0—\mathrm{H}_0)/4\pi]$ is the mean maximum of intensity of magnetisation at the centre of the wire, and $2l$ is the length of the wire.

Initially h_0 is nearly proportional to B_0. Since, however, much of the induction leaks out from the wire by the cylindrical surface, thus developing a distribution of magnetism along the wire, h_0 is considerably greater than it would be were the "poles" at the ends of the wire. As B_0 increases, h_0 increases less rapidly, reaches a maximum for a value of B_0 [10300] somewhat greater than B'_0 [8400], the value corresponding

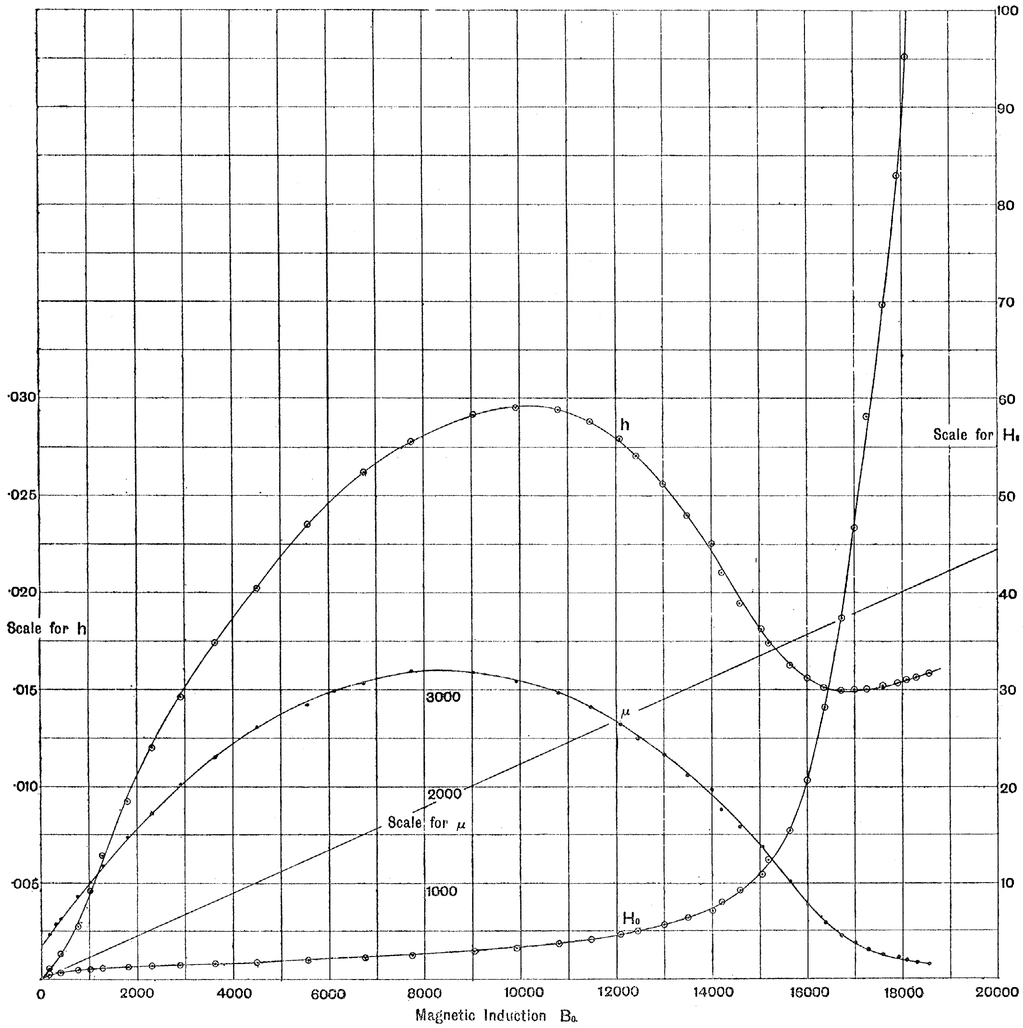

Fig. 18.

to μ' the maximum value of μ, and then diminishes. As B_0 increases still further, h_0 diminishes rapidly till B_0 reaches the stage where μ begins to diminish comparatively slowly. After passing a minimum h_0 again increases, and in this last stage is less than $2I_0A/l^2$.

§ 76. The experiments of Mr. C. G. Lamb, "On the Distribution of Magnetic Induction in a Long Iron Bar,"* serve to explain the rise of h_0 to a maximum, and its subsequent fall. For he found that as B_0 increases from zero, the percentage of the induction at the centre of the rod which leaks out between the centre and the end of

* 'Proc. Phys. Soc.,' vol. 16, p. 509; or 'Phil. Mag.,' Sept., 1899.

the rod, increases until B_0 reaches B'_0 approximately. Thus as B_0 increases we may regard the "poles" as moving inwards towards the centre. There is thus a double reason why h_0 should increase with B_0 during this stage. When B_0 increases beyond B'_0, he found that the "poles" move out again towards the ends of the rod, and we must suppose that this motion more than compensates for the increasing strength of the poles with increasing B_0.*

In our experiments, for the largest values of B_0, h_0 was *less* than $2I_0A/l^2$. This effect can only arise from the development of a subsidiary pole between the centre and either end of the wire, with sign opposite to that of the pole at the end. This result is easily seen to be impossible with a bar of uniform material as long as the induction is a definite single-valued function of the magnetic force, which increases as the magnetic force increases. It must therefore be due to the effects of hysteresis, which becomes an increasingly important factor as the point of maximum permeability is reached and passed. A different result might perhaps have been obtained if the magnetic force had been gradually increased from zero. But in our experiments, as in those of LAMB, the magnetic force was put through several cycles between the limits $\pm H_0$ before the observations were made.

§ 77. When iron is tested by the ballistic method for hysteresis, the specimen is subjected to an applied magnetic force which is reversed between the limits $\pm H_0$ until the iron has reached a cyclic state. This attained, the points on the cyclic B—H diagram are found from the throws of the galvanometer which occur when the magnetic force is suddenly changed, by means of a special key, from H_0 to a series of values between H_0 and $-H_0$.

We made hysteresis tests by this method upon two specimens, determining also the demagnetising force, h, at the centre of the wire, for each value of H which was employed in constructing the B—H curve. Thus, the magnetic force was changed from H_0 to H, and the throw of the galvanometer, giving $B_0 - B$, was observed, and then, without altering the magnetic force from its value H, the deflexion of the magnetometer was noted. We reckon h_0 positive when its direction is opposite to that of the induction at the centre of the specimen.

§ 78. The first specimen was a bundle of ten iron wires with a total area of section of ·0412 sq. centim. The length of the wire was equal to the length of the solenoid,

* Dr. L. HOLBORN, in a paper "On the Distribution of Induced Magnetism in Cylinders" ('Sitzungsberichte der Akademie der Wissenschaften zu Berlin,' 17th February, 1898), has obtained a result similar to that found by Mr. LAMB. He used two secondary coils, one a uniformly wound solenoid closely fitting the rod, the other a coil wound about the centre of the rod. The rod and the secondary coils were placed inside a long magnetising solenoid. By comparing the changes of induction through these two secondaries due to a reversal of the primary current, he found the distance λ between the "centres of gravity" of the free magnetism on the two halves of the rod. He found that the "centres of gravity" move towards the centre as B_0 increases, until B_0 reaches B'_0. A further increase of B_0 caused the "centres of gravity" to move out again from the centre. Dr. HOLBORN made similar experiments on ellipsoids, and found that for them λ was remarkably constant.

47 centims., and thus since the internal radius of the windings was about 2 centims. the applied magnetic force was far from uniform near the ends of the wires. The results of this experiment are shown in figs. 19, 20. As we pointed out in § 18, h exhibits hysteresis with respect to both B and H. In fig. 19 the straight line

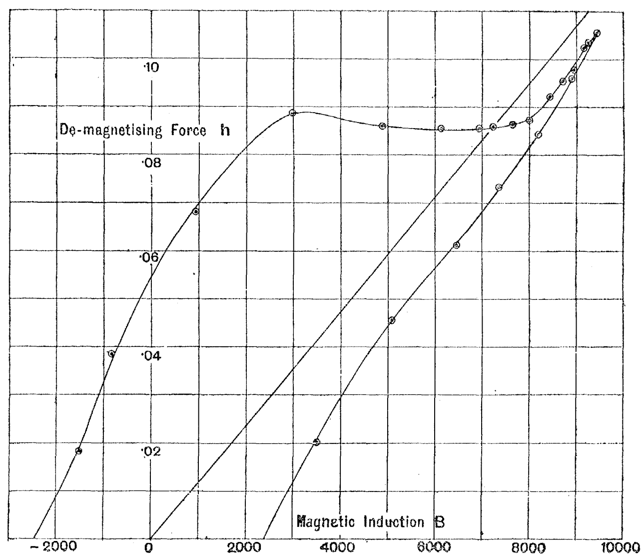

Fig. 19.

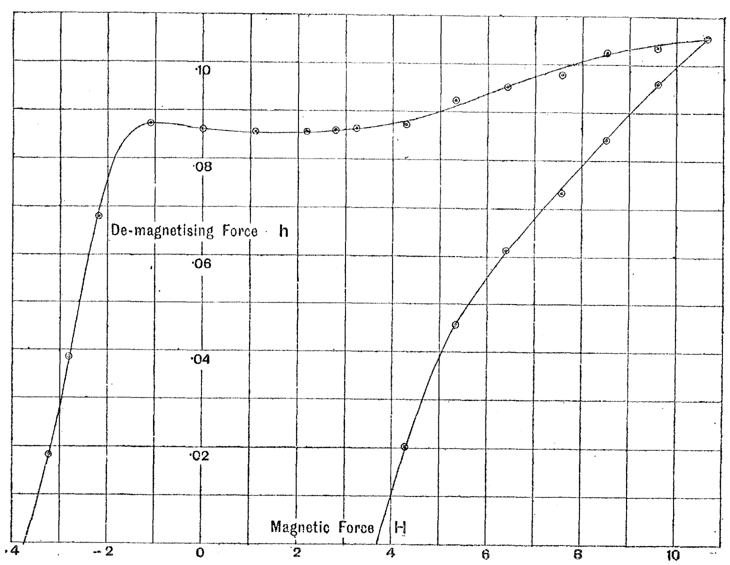

Fig. 20.

through the origin represents $2BA/4\pi l^2$, the demagnetising force corresponding to uniform magnetisation, H in these experiments being negligible in comparison with B. From the areas of the curves we found

$$\int H dB = 89200, \qquad \int h dB = 949, \qquad \int h dH = 1{\cdot}57.$$

§ 79. The second specimen was a single iron wire 47 centims. in length, and ·00412 sq. centim. in section. The magnetising solenoid was that described in § 75, and thus the applied magnetic force was practically uniform over the whole length of the wire. We examined the polarity of the eastern end of the wire at every stage by means of a small compass. We show the results of the experiment in figs. 21, 22, and also in the following table, where the last column indicates the

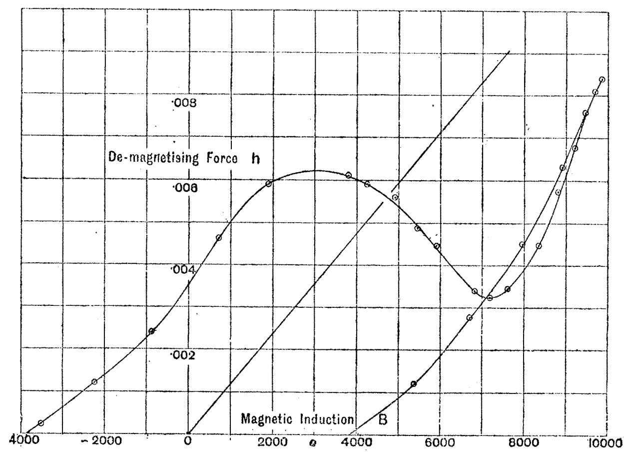

Fig. 21.

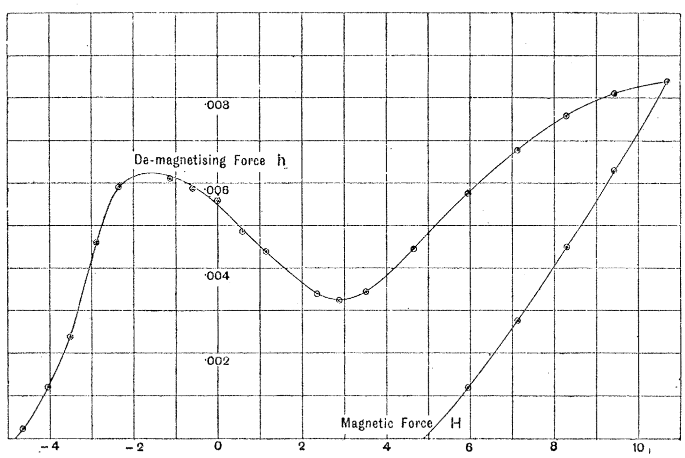

Fig. 22.

polarity of the eastern end. It will be seen that the polarity of the end of the wire agrees in every case with that corresponding to the direction of the induction at the centre of the wire. In fig. 21 the straight line through the origin represents $2BA/4\pi l^2$, the demagnetising force corresponding to uniform magnetisation, H in these experiments being negligible in comparison with B.

H.	B.	h.	Pole.	H.	B.	h.	Pole.
10·67	9850	·00839	N.	– 0·59	4250	·00587	N.
9·44	9675	·00810	N.	– 1·13	3800	·00610	N.
8·30	9465	·00757	N.	– 2·36	1900	·00590	N.
7·12	9204	·00676	N.	– 2·88	730	·00460	N.
5·95	8800	·00575	N.	– 3·52	– 850	·00238	S.
4·65	8320	·00446	N.	– 4·05	– 2200	·00120	S.
3·52	7590	·00344	N.	– 4·65	– 3480	·00023	S.
2·88	7190	·00323	N.	– 5·95	– 5370	– ·00119	S.
2·36	6820	·00338	N.	– 7·12	– 6680	– ·00276	S.
1·13	5900	·00440	N.	– 8·30	– 7950	– ·00450	S.
0·59	5450	·00486	N.	– 9·44	– 8900	– ·00630	S.
0	4900	·00559	N.	– 10·67	– 9850	– ·00839	S.

As B diminishes from 9850, h diminishes to a minimum at B = 7200, and then increases to a maximum at B = 3000; after this it passes to its greatest negative value corresponding to B = −9850 without passing through a maximum or minimum. As B increases again to 9850, h goes through a similar set of changes, its values from B = 7000 to 9850 differing but little from those which it had when B was diminishing. From B = 9850 to 4700, h is less than $2BA/4\pi l^2$, and from B = 0 to B = − 3800, h has the opposite sign to B. Both these cases require that there should be "poles" on the wire between the centre and the ends, with signs opposite to those of the poles at the ends of the wire. Our arrangement of apparatus was not well adapted for detecting these subsidiary poles, since the magnetic force due to a uniformly distributed pole would be at right angles to the wire, and therefore parallel to the magnetic meridian, thus producing no deflexion of the search compass. Still we were able to verify the conclusion when B was −3480, for though the corresponding S-magnetism appeared at the eastern portion of the wire, being most concentrated at a point about 7 centims. from the end, yet we found a weak N-pole at 9 centims. from the same end.

From the areas of the curves we found

$$\int H dB = 102100, \qquad \int h dB = 72{\cdot}3, \qquad \int h dH = {\cdot}118.$$

These experiments add emphasis to Mr. LAMB's remark that "the magnetometric method [of determining I—H curves], although extremely useful for comparative work, must be used with much caution in determinations of an absolute character."

[*December* 23, 1901.—C. BENEDICKS ('Annalen der Physik,' 1901, vol. 6, p. 726) compared the B—H curve for a cylinder of steel with the curve for an ellipsoid formed out of the same piece of metal, and deduces that, as B increases, h rises to a maximum and then rapidly decreases, as in our fig. 18. From EWING's experiments ('Phil. Trans.,' 1885, pp. 532, 535) on (1) a ring, (2) straight pieces of iron wire, he deduces the same result. Neither experimenter, however, reached the minimum of h.]